

Global Well-posedness in Critical Besov Spaces for Two-fluid Euler-Maxwell Equations

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Abstract

In this paper, we study the well-posedness in critical Besov spaces for two-fluid Euler-Maxwell equations, which is different from the one-fluid case. We need to deal with the difficulties mainly caused by the nonlinear coupling and cancelation between two carriers. Precisely, we first obtain the local existence and blow-up criterion of classical solutions to the Cauchy problem and periodic problem pertaining to data in Besov spaces with critical regularity. Furthermore, we construct the global existence of classical solutions with aid of a different energy estimate (in comparison with the one-fluid case) provided the initial data is small under certain norms. Finally, we establish the large-time asymptotic behavior of global solutions near equilibrium in Besov spaces with relatively lower regularity.

Keywords. two-fluid Euler-Maxwell equations, classical solutions, Chemin-Lerner spaces

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1 Introduction

As an un-magnetized plasma is operated under some high frequency conditions (such as photo-conductive switches, electro-optics and high-speed computers, etc.), electromagnetic fields are generated by moving electrons and ions, then the two carriers transport interacts with the propagating magnetic waves. In this case, the transport process is typically governed by Euler-Maxwell equations, which take the form of Euler equations for the conservation laws of mass density and current density for carriers, coupled to Maxwell's equations for self-consistent electromagnetic fields. By some appropriate re-scaling, the two-fluid compressible Euler-Maxwell equations are written, in nondimensional form, as (see, e.g., [14])

$$\begin{cases} \partial_t n_{\pm} + \nabla \cdot (n_{\pm} u_{\pm}) = 0, \\ \partial_t (n_{\pm} u_{\pm}) + \nabla \cdot (n_{\pm} u_{\pm} \otimes u_{\pm}) + \nabla p_{\pm}(n_{\pm}) = \mp n_{\pm} (E + \varepsilon u_{\pm} \times B) - n_{\pm} u_{\pm} / \tau_{\pm}, \\ \varepsilon \lambda^2 \partial_t E - \nabla \times B = \varepsilon (n_+ u_+ - n_- u_-), \\ \varepsilon \partial_t B + \nabla \times E = 0, \\ \lambda^2 \nabla \cdot E = n_- - n_+, \quad \nabla \cdot B = 0. \end{cases} \quad (1.1)$$

for $(t, x) \in [0, +\infty) \times \Omega$ ($\Omega = \mathbb{R}^N$ or \mathbb{T}^N , $N = 2, 3$). Here the unknowns $n_{\pm} = n_{\pm}(t, x) > 0$, $u_{\pm} = u_{\pm}(t, x) \in \Omega$, respectively, stand for densities and velocities of the electrons (+) and ions (-). $E = E(t, x) \in \Omega$ and $B = B(t, x) \in \Omega$ denote the electric field and magnetic field, respectively. The pressure functions $p_{\pm}(\cdot)$ satisfy the usual γ -law: $p_{\pm}(n_{\pm}) = A_{\pm} n_{\pm}^{\gamma}$, where $A_{\pm} > 0$ are some physical constants and the adiabatic exponent $\gamma \geq 1$. τ_{\pm} are the (scaled) constants for the momentum-relaxation timed of electrons and ions, and $\lambda > 0$ is the Debye length. $c = (\epsilon_0 v_0)^{-\frac{1}{2}} > 0$ is the speed of light, where ϵ_0 and v_0 are the vacuum permittivity and permeability. Setting $\varepsilon = \frac{1}{c}$. The parameters τ_{\pm}, λ and ε arising from nondimensionalization are independent each other, and they are assumed to be very small compared to the reference physical size. In this paper, we set these physical constants to be one.

It is not difficult to see below in the text that the Euler-Maxwell equations (1.1) consist of a quasi-linear symmetrizable hyperbolic system, the main feature of which is the finite time blow-up of classical solutions even when the initial data are smooth and small. Hence, the qualitative study and device simulation of (1.1) are far from trivial. The primary objective of this paper is to establish the global well-posedness for the corresponding Cauchy problem and periodic problem. For this purpose, (1.1) is equipped with the following initial data

$$(n_{\pm}, u_{\pm}, E, B)(x, 0) = (n_{\pm 0}, u_{\pm 0}, E_0, B_0)(x) \quad (1.2)$$

satisfying the compatible conditions

$$\nabla \cdot E_0 = n_{-0} - n_{+0}, \quad \nabla \cdot B_0 = 0, \quad x \in \Omega. \quad (1.3)$$

In the past years, the Euler-Maxwell equations have attached much attention. In one space dimension, using the Godunov scheme with the fractional step together with the compensated compactness theory, Chen, Jerome and Wang [4] constructed the existence of a global weak solution to the initial boundary value problem for arbitrarily large initial data in L^{∞} . Assuming initial data in Sobolev spaces $H^s(\mathbb{R}^3)$ with higher regularity $s > 5/2$, a local existence theory of smooth solutions for the Cauchy problem of Euler-Maxwell equations was established in [10]

by the author's modification of the classical semigroup-resolvent approach of Kato [11]. Subsequently, the global existence and the large time behavior of smooth solutions with small perturbations were obtained by Peng, Wang and Gu [18], Duan [6, 7], Ueda, Wang and Kawashima [21, 22]. In addition, the asymptotic limits such as the non-relativistic limit ($\varepsilon \rightarrow 0$), the quasi-neutral limit ($\lambda \rightarrow 0$) and the combined non-relativistic and quasi-neutral limits ($\varepsilon = \lambda \rightarrow 0$) have been justified by Peng and Wang [15, 16, 17]. The reader is also referred to [25] for combined diffusive relaxation limits and [19, 20] for WKB asymptotics; and references therein.

Up to now, the study for Euler-Maxwell equations in several dimensions are still far from well known in the framework of critical spaces. Recently, using the low- and high-frequency decomposition arguments, we constructed uniform (global) classical solutions (around constant equilibrium) to the Cauchy problem of one-fluid Euler-Maxwell system in Chemin-Lerner spaces with critical regularity. Furthermore, based on the Aubin-Lions compactness lemma, it is justified that the (scaled) classical solutions converge globally in time to the solutions of compressible Euler-Poisson equations in the process of nonrelativistic limit and to that of drift-diffusion equations under the relaxation limit or the combined nonrelativistic and relaxation limits, see [23].

In the present paper, we extend those results in [23] to the two-fluid Euler-Maxwell equations (1.1). More precisely, we consider the perturbation near the constant equilibrium state $(1, 0, 1, 0, 0, \bar{B})(\bar{B} \in \Omega)$ which is a particular solution of the system (1.1)-(1.2), and achieve local well-posedness for general data and global well-posedness for small data. It should be pointed out that (1.1) is different from the one-fluid case and this extension is not trivial. We are faced with new difficulties arising from the more complicated nonlinear coupling and cancelation between two carriers. For instance, the expected dissipation rates for the densities of electrons and ions are absent in whole space \mathbb{R}^N , and we only capture the *weaker* dissipation ones from contributions of $(\nabla n_+, \nabla n_-)$ and $n_+ - n_-$. Therefore, in order to close the “*a priori*” estimates in critical spaces, new techniques in comparison with [23] are adopted. Indeed, we perform the *homogeneous* blocks rather than the *inhomogeneous* blocks to localize the symmetric system, as one captures the dissipation rate for velocities. Furthermore, the elementary fact established in the recent work [24], which indicates the relations between homogeneous Chemin-Lerner spaces and inhomogeneous Chemin-Lerner spaces, will be used. In addition, different from that in [23], we modify the nonlinear smooth function arising from the symmetrization a little such that $h(0) = h'(0) = 0$, then we take full advantage of the continuity for compositions in space-time Besov spaces (Chemin-Lerner spaces) which is a natural generalization from Besov spaces to Chemin-Lerner spaces, to estimate the cancelation of densities between two carriers effectively. For above details, see Sect. 3, Lemma 4.1-4.2 and Proposition 5.1-5.2.

To state main results more explicitly, we first introduce the functional spaces

$$\tilde{\mathcal{C}}_T(B_{p,r}^s(\Omega)) := \tilde{L}_T^\infty(B_{p,r}^s(\Omega)) \cap \mathcal{C}([0, T], B_{p,r}^s(\Omega))$$

and

$$\tilde{\mathcal{C}}_T^1(B_{p,r}^s(\Omega)) := \{f \in \mathcal{C}^1([0, T], B_{p,r}^s(\Omega)) | \partial_t f \in \tilde{L}_T^\infty(B_{p,r}^s(\Omega))\},$$

where the index $T > 0$ will be omitted when $T = +\infty$, the reader is referred to Definition 2.1 below for Chemin-Lerner spaces.

Throughout this paper, let us denote by s_c the critical number $1 + N/2$. First of all, we give the local existence and blow-up criterion of classical solutions to (1.1)-(1.2) away from the vacuum.

Theorem 1.1. *Let $\bar{B} \in \Omega$ be any given constant. Suppose that $n_{\pm 0} - 1, u_{\pm 0}, E_0$ and $B_0 - \bar{B} \in B_{2,1}^{s_c}(\Omega)$ satisfy $n_{\pm 0} > 0$ and the compatible conditions (1.3). Then there exists a time $T_0 > 0$ such that*

- (i) *Existence: the system (1.1)-(1.2) has a unique solution $(n_{\pm}, u_{\pm}, E, B) \in \mathcal{C}^1([0, T_0] \times \Omega)$ with $n_{\pm} > 0$ for all $t \in [0, T_0]$ and $(n_{\pm} - 1, u_{\pm}, E, B - \bar{B}) \in \tilde{\mathcal{C}}_{T_0}(B_{2,1}^{s_c}(\Omega)) \cap \tilde{\mathcal{C}}_{T_0}^1(B_{2,1}^{s_c-1}(\Omega))$;*
- (ii) *Blow-up criterion: if the maximal time $T^*(> T_0)$ of existence of such a solution is finite, then*

$$\limsup_{t \rightarrow T^*} \|n_{\pm}(t, \cdot) - 1, u_{\pm}(t, \cdot), E(t, \cdot), B(t, \cdot) - \bar{B}\|_{B_{2,1}^{s_c}(\Omega)} = \infty$$

if and only if

$$\int_0^{T^*} \|(\nabla n_{\pm}, \nabla u_{\pm}, \nabla E, \nabla B)(t, \cdot)\|_{L^\infty(\Omega)} dt = \infty.$$

Remark 1.1. Recently, Xu and Kawashima [24] have established a general theory on the well-posedness of generally symmetrizable hyperbolic systems in the framework of critical Chemin-Lerner spaces, which is regarded as the generalization of the classical local existence theory of Kato and Majda [11, 12]. As a matter of fact, the results are also adapted to the periodic case. As in Sect. 3, we see that (1.1) is transformed into a symmetric hyperbolic system equivalently. Hence, the general theory can be applied to the Euler-Maxwell equations. It is worth noting that the blow-up criterion of classical solutions to the Euler-Maxwell equations is obtained firstly in the present paper.

In small amplitude regime, we establish the following global well-posedness to (1.1)-(1.2) in critical spaces.

Theorem 1.2. *Let $\bar{B} \in \Omega$ be any given constant. Suppose that $(n_{\pm 0} - 1, u_{\pm 0}, E_0, B_0 - \bar{B}) \in B_{2,1}^{s_c}(\Omega)$ satisfy the compatible conditions (1.3). There exists a positive constant δ_0 such that if*

$$\|(n_{\pm 0} - 1, u_{\pm 0}, E_0, B_0 - \bar{B})\|_{B_{2,1}^{s_c}(\Omega)} \leq \delta_0,$$

then the system (1.1)-(1.2) admits a unique global solution (n_{\pm}, u_{\pm}, E, B) satisfying

$$(n_{\pm}, u_{\pm}, E, B) \in \mathcal{C}^1([0, \infty) \times \Omega)$$

and

$$(n_{\pm} - 1, u_{\pm}, E, B - \bar{B}) \in \tilde{\mathcal{C}}(B_{2,1}^{s_c}(\Omega)) \cap \tilde{\mathcal{C}}^1(B_{2,1}^{s_c-1}(\Omega)).$$

Moreover, there are two positive constants μ_0 and C_0 such that

- (i) *when $\Omega = \mathbb{R}^N$, it yields the following*

$$\begin{aligned} & \|(n_{\pm} - 1, u_{\pm}, E, B - \bar{B})\|_{\tilde{L}^\infty(B_{2,1}^{s_c}(\Omega))} \\ & + \mu_0 \left\{ \|(n_+ - n_-, u_{\pm})\|_{\tilde{L}^2(B_{2,1}^{s_c}(\Omega))} + \|(\nabla n_{\pm}, E)\|_{\tilde{L}^2(B_{2,1}^{s_c-1}(\Omega))} + \|\nabla B\|_{\tilde{L}^2(B_{2,1}^{s_c-2}(\Omega))} \right\} \\ & \leq C_0 \|(n_{\pm 0} - 1, u_{\pm 0}, E_0, B_0 - \bar{B})\|_{B_{2,1}^{s_c}(\Omega)}; \end{aligned} \tag{1.4}$$

(ii) when $\Omega = \mathbb{T}^N$, we further set $\bar{n}_{\pm 0} = 1$, it yields the following

$$\begin{aligned} & \| (n_{\pm} - 1, u_{\pm}, E, B - \bar{B}) \|_{\tilde{L}^{\infty}(B_{2,1}^{s_c}(\Omega))} \\ & + \mu_0 \left\{ \| (n_{\pm} - 1, u_{\pm}) \|_{\tilde{L}^2(B_{2,1}^{s_c}(\Omega))} + \| E \|_{\tilde{L}^2(B_{2,1}^{s_c-1}(\Omega))} + \| \nabla B \|_{\tilde{L}^2(B_{2,1}^{s_c-2}(\Omega))} \right\} \\ & \leq C_0 \| (n_{\pm 0} - 1, u_{\pm 0}, E_0, B_0 - \bar{B}) \|_{B_{2,1}^{s_c}(\Omega)}, \end{aligned} \quad (1.5)$$

where \bar{f} denotes the mean value of $f(x)$ over \mathbb{T}^N , that is,

$$\bar{f} = \frac{1}{|\mathbb{T}^N|} \int_{\mathbb{T}^N} f(x) dx.$$

Remark 1.2. Following from approaches in the current paper, the well-posedness results to the Cauchy problem and periodic problem pertaining to data in the *supercritical* Besov spaces $B_{p,r}^s(\Omega)$ ($s > s_c, p = 2, 1 \leq r \leq \infty$) can be also established. Furthermore, the fact that Sobolev spaces $H^s(\Omega) := B_{2,2}^s(\Omega)$ allows the results to be also true in the usual Sobolev spaces with $s > s_c$.

Remark 1.3. In the whole space, the energy inequality (1.4) is not so surprising in comparison with the one-fluid case as in [23], however, it is indeed *different*. Due to the nonlinear coupling and cancelation between two carriers, the dissipation rates of (n_+, n_-) does not appear in (1.4) any more, and the dissipation rates from $n_+ - n_-$ and $(\nabla n_+, \nabla n_-)$ are available only. In this case, to overcome the technical difficulties occurring in the *a priori* estimates, some useful facts in Chemin-Lerner spaces are developed. It is worth noting that the dissipation rate of (n_+, n_-) itself in the periodic case can be obtained, see the proof of Theorem 1.2. Besides, from (1.4)-(1.5), we see that there is a “1-regularity-loss” phenomenon for the dissipation rates of electromagnetic field (E, B) .

As a direct consequence of Theorem 1.2, we obtain the large-time asymptotic behavior of global solutions near the equilibrium state $(1, 0, 1, 0, 0, \bar{B})$ in some Besov spaces.

Corollary 1.1. *Let (n_{\pm}, u_{\pm}, E, B) be the global-in-time solution in Theorem 1.2, it holds that ($\varepsilon > 0$)*

$$\| n_+(t, \cdot) - n_-(t, \cdot), u_{\pm}(t, \cdot) \|_{B_{2,1}^{s_c-\varepsilon}(\Omega)} \rightarrow 0,$$

$$\| E(t, \cdot) \|_{B_{2,1}^{s_c-1-\varepsilon}(\Omega)} \rightarrow 0, \quad \| B(t, \cdot) - \bar{B} \|_{B_{p,1}^{s_c-2-\varepsilon}(\Omega)} \rightarrow 0,$$

moreover,

$$\| n_{\pm}(t, \cdot) - 1 \|_{B_{p,1}^{s_c-1-\varepsilon}(\mathbb{R}^N)} \rightarrow 0 \quad \left(p = \frac{2N}{N-2}, N > 2 \right),$$

$$\| n_{\pm}(t, \cdot) - 1 \|_{B_{2,1}^{s_c-\varepsilon}(\mathbb{T}^N)} \rightarrow 0,$$

as the time variable $t \rightarrow +\infty$.

Remark 1.4. Recalling the Corollary 5.1 in [9], we omit details of the proof of Corollary 1.1, since they are similarly followed by the Gagliardo-Nirenberg-Sobolev inequality (see, e.g., [8]) and interpolation arguments. In addition, from the embedding $B_{2,1}^{s_c-\varepsilon} \hookrightarrow B_{p,1}^{s_c-1-\varepsilon}$ ($N = 3$), we know that the large-time asymptotic behavior of densities n_{\pm} in the whole space case is *weaker* than that in the periodic case.

Remark 1.5. Following from the similar manners, the corresponding results can be obtained for non-isentropic two-fluid Euler-Maxwell equations, which include the temperature transport equations of carriers rather than the assumed pressure-density relations as in (1.1) only. Let us mention that the dissipation rates of temperatures will behave as that of velocities, that is, there is no regularity-loss phenomenon for temperatures.

The rest of this paper unfolds as follows. In Sect. 2, we briefly review some useful properties on Besov spaces. In Sect. 3, we establish the local existence and blow-up criterion for the Euler-Maxwell equations (1.1). Sect. 4 is devoted to the global existence of classical solutions in critical spaces. In the last section (Sect. 5), we remark a natural generalization on the continuity of composition functions in Chemin-Lerner spaces.

2 Littlewood-Paley theory and functional spaces

Throughout the paper, $f \lesssim g$ denotes $f \leq Cg$, where $C > 0$ is a generic constant. $f \approx g$ means $f \lesssim g$ and $g \lesssim f$. Denote by $\mathcal{C}([0, T], X)$ (resp., $\mathcal{C}^1([0, T], X)$) the space of continuous (resp., continuously differentiable) functions on $[0, T]$ with values in a Banach space X . Also, $\|(f, g, h)\|_X$ means $\|f\|_X + \|g\|_X + \|h\|_X$, where $f, g, h \in X$. $\langle f, g \rangle$ denotes the inner product of two functions f, g in $L^2(\mathbb{R}^N)$.

In this section, we briefly review the Littlewood-Paley decomposition and some properties of Besov spaces. The reader is also referred to, *e.g.*, [2, 5] for more details.

Let us start with the Fourier transform. The Fourier transform \hat{f} of a L^1 -function f is given by

$$\mathcal{F}f = \int_{\mathbb{R}^N} f(x) e^{-2\pi i x \cdot \xi} dx.$$

More generally, the Fourier transform of any $f \in \mathcal{S}'$, the space of tempered distributions, is given by

$$(\mathcal{F}f, g) = (f, \mathcal{F}g)$$

for any $g \in \mathcal{S}$, the Schwartz class.

First, we fix some notation.

$$\mathcal{S}_0 = \left\{ \phi \in \mathcal{S}, \partial^\alpha \mathcal{F}f(0) = 0, \forall \alpha \in \mathbb{N}^N \text{ multi-index} \right\}.$$

Its dual is given by

$$\mathcal{S}'_0 = \mathcal{S}' / \mathcal{P},$$

where \mathcal{P} is the space of polynomials.

We now introduce a dyadic partition of \mathbb{R}^N . We choose $\phi_0 \in \mathcal{S}$ such that ϕ_0 is even,

$$\text{supp} \phi_0 := A_0 = \left\{ \xi \in \mathbb{R}^N : \frac{3}{4} \leq |\xi| \leq \frac{8}{3} \right\}, \text{ and } \phi_0 > 0 \text{ on } A_0.$$

Set $A_q = 2^q A_0$ for $q \in \mathbb{Z}$. Furthermore, we define

$$\phi_q(\xi) = \phi_0(2^{-q}\xi)$$

and define $\Phi_q \in \mathcal{S}$ by

$$\mathcal{F}\Phi_q(\xi) = \frac{\phi_q(\xi)}{\sum_{q \in \mathbb{Z}} \phi_q(\xi)}.$$

It follows that both $\mathcal{F}\Phi_q(\xi)$ and Φ_q are even and satisfy the following properties:

$$\mathcal{F}\Phi_q(\xi) = \mathcal{F}\Phi_0(2^{-q}\xi), \quad \text{supp } \mathcal{F}\Phi_q(\xi) \subset A_q, \quad \Phi_q(x) = 2^{qN}\Phi_0(2^q x)$$

and

$$\sum_{q=-\infty}^{\infty} \mathcal{F}\Phi_q(\xi) = \begin{cases} 1, & \text{if } \xi \in \mathbb{R}^N \setminus \{0\}, \\ 0, & \text{if } \xi = 0. \end{cases}$$

As a consequence, for any $f \in S'_0$, we have

$$\sum_{q=-\infty}^{\infty} \Phi_q * f = f.$$

To define the homogeneous Besov spaces, we set

$$\dot{\Delta}_q f = \Phi_q * f, \quad q = 0, \pm 1, \pm 2, \dots$$

Definition 2.1. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the homogeneous Besov spaces $\dot{B}_{p,r}^s$ is defined by

$$\dot{B}_{p,r}^s = \{f \in S'_0 : \|f\|_{\dot{B}_{p,r}^s} < \infty\},$$

where

$$\|f\|_{\dot{B}_{p,r}^s} = \begin{cases} \left(\sum_{q \in \mathbb{Z}} (2^{qs} \|\dot{\Delta}_q f\|_{L^p})^r \right)^{1/r}, & r < \infty, \\ \sup_{q \in \mathbb{Z}} 2^{qs} \|\dot{\Delta}_q f\|_{L^p}, & r = \infty. \end{cases}$$

To define the inhomogeneous Besov spaces, we set $\Psi \in \mathcal{C}_0^\infty(\mathbb{R}^N)$ be even and satisfy

$$\mathcal{F}\Psi(\xi) = 1 - \sum_{q=0}^{\infty} \mathcal{F}\Phi_q(\xi).$$

It is clear that for any $f \in S'$, yields

$$\Psi * f + \sum_{q=0}^{\infty} \Phi_q * f = f.$$

We further set

$$\Delta_q f = \begin{cases} 0, & j \leq -2, \\ \Psi * f, & j = -1, \\ \Phi_q * f, & j = 0, 1, 2, \dots \end{cases}$$

Definition 2.2. For $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, the inhomogeneous Besov spaces $B_{p,r}^s$ is defined by

$$B_{p,r}^s = \{f \in S' : \|f\|_{B_{p,r}^s} < \infty\},$$

where

$$\|f\|_{B_{p,r}^s} = \begin{cases} \left(\sum_{q=-1}^{\infty} (2^{qs} \|\Delta_q f\|_{L^p})^r \right)^{1/r}, & r < \infty, \\ \sup_{q \geq -1} 2^{qs} \|\Delta_q f\|_{L^p}, & r = \infty. \end{cases}$$

Let us point out that the definitions of $\dot{B}_{p,r}^s$ and $B_{p,r}^s$ does not depend on the choice of the Littlewood-Paley decomposition. Now, we state some basic properties, which will be used in subsequent analysis.

Lemma 2.1. (*Bernstein inequality*) *Let $k \in \mathbb{N}$ and $0 < R_1 < R_2$. There exists a constant C , depending only on R_1, R_2 and N , such that for all $1 \leq a \leq b \leq \infty$ and $f \in L^a$,*

$$\text{Supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^N : |\xi| \leq R_1 \lambda\} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^b} \leq C^{k+1} \lambda^{k+d(\frac{1}{a}-\frac{1}{b})} \|f\|_{L^a};$$

$$\text{Supp } \mathcal{F}f \subset \{\xi \in \mathbb{R}^N : R_1 \lambda \leq |\xi| \leq R_2 \lambda\} \Rightarrow C^{-k-1} \lambda^k \|f\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha f\|_{L^a} \leq C^{k+1} \lambda^k \|f\|_{L^a}.$$

As a direct corollary of the above inequality, we have

Remark 2.1. For all multi-index α , it holds that

$$\begin{aligned} \frac{1}{C} \|f\|_{\dot{B}_{p,r}^{s+|\alpha|}} &\leq \|\partial^\alpha f\|_{\dot{B}_{p,r}^s} \leq C \|f\|_{\dot{B}_{p,r}^{s+|\alpha|}}; \\ \|\partial^\alpha f\|_{B_{p,r}^s} &\leq C \|f\|_{B_{p,r}^{s+|\alpha|}}. \end{aligned}$$

The second one is the embedding properties in Besov spaces.

Lemma 2.2. *Let $s \in \mathbb{R}$ and $1 \leq p, r \leq \infty$, then*

- (1) *If $s > 0$, then $B_{p,r}^s = L^p \cap \dot{B}_{p,r}^s$;*
- (2) *If $\tilde{s} \leq s$, then $B_{p,r}^s \hookrightarrow B_{p,\tilde{r}}^{\tilde{s}}$;*
- (3) *If $1 \leq r \leq \tilde{r} \leq \infty$, then $\dot{B}_{p,r}^s \hookrightarrow \dot{B}_{p,\tilde{r}}^s$ and $B_{p,r}^s \hookrightarrow B_{p,\tilde{r}}^s$;*
- (4) *If $1 \leq p \leq \tilde{p} \leq \infty$, then $\dot{B}_{p,r}^s \hookrightarrow \dot{B}_{\tilde{p},r}^{s-N(\frac{1}{p}-\frac{1}{\tilde{p}})}$ and $B_{p,r}^s \hookrightarrow B_{\tilde{p},r}^{s-N(\frac{1}{p}-\frac{1}{\tilde{p}})}$;*
- (5) *$\dot{B}_{p,1}^{N/p} \hookrightarrow \mathcal{C}_0$, $B_{p,1}^{N/p} \hookrightarrow \mathcal{C}_0$ ($1 \leq p < \infty$);*

where \mathcal{C}_0 is the space of continuous bounded functions which decay at infinity.

On the other hand, we also present the definition of Chemin-Lerner space-time spaces first introduced by J.-Y. Chemin and N. Lerner [3], which are the refinement of the spaces $L_T^\theta(\dot{B}_{p,r}^s)$ or $L_T^\theta(B_{p,r}^s)$.

Definition 2.3. *For $T > 0, s \in \mathbb{R}, 1 \leq r, \theta \leq \infty$, the homogeneous mixed time-space Besov spaces $\tilde{L}_T^\theta(\dot{B}_{p,r}^s)$ is defined by*

$$\tilde{L}_T^\theta(\dot{B}_{p,r}^s) := \{f \in L^\theta(0, T; \mathcal{S}'_0) : \|f\|_{\tilde{L}_T^\theta(\dot{B}_{p,r}^s)} < +\infty\},$$

where

$$\|f\|_{\tilde{L}_T^\theta(\dot{B}_{p,r}^s)} := \left(\sum_{q \in \mathbb{Z}} (2^{qs} \|\Delta_q f\|_{L_T^\theta(L^p)})^r \right)^{\frac{1}{r}}$$

with the usual convention if $r = \infty$.

Definition 2.4. For $T > 0, s \in \mathbb{R}, 1 \leq r, \theta \leq \infty$, the inhomogeneous mixed time-space Besov spaces $\tilde{L}_T^\theta(B_{p,r}^s)$ is defined by

$$\tilde{L}_T^\theta(B_{p,r}^s) := \{f \in L^\theta(0, T; \mathcal{S}') : \|f\|_{\tilde{L}_T^\theta(B_{p,r}^s)} < +\infty\},$$

where

$$\|f\|_{\tilde{L}_T^\theta(B_{p,r}^s)} := \left(\sum_{q \geq -1} (2^{qs} \|\Delta_q f\|_{L_T^\theta(L^p)})^r \right)^{\frac{1}{r}}$$

with the usual convention if $r = \infty$.

Next we state some basic properties on the inhomogeneous Chemin-Lerner spaces only, since the similar ones are true in the homogeneous Chemin-Lerner spaces.

The first one is that $\tilde{L}_T^\theta(B_{p,r}^s)$ may be linked with the classical spaces $L_T^\theta(B_{p,r}^s)$ via the Minkowski's inequality:

Remark 2.2. It holds that

$$\|f\|_{\tilde{L}_T^\theta(B_{p,r}^s)} \leq \|f\|_{L_T^\theta(B_{p,r}^s)} \text{ if } r \geq \theta; \quad \|f\|_{\tilde{L}_T^\theta(B_{p,r}^s)} \geq \|f\|_{L_T^\theta(B_{p,r}^s)} \text{ if } r \leq \theta.$$

Let us also recall the property of continuity for product in Chemin-Lerner spaces $\tilde{L}_T^\theta(B_{p,r}^s)$.

Proposition 2.1. *The following inequality holds:*

$$\|fg\|_{\tilde{L}_T^\theta(B_{p,r}^s)} \leq C(\|f\|_{L_T^{\theta_1}(L^\infty)} \|g\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)} + \|g\|_{L_T^{\theta_3}(L^\infty)} \|f\|_{\tilde{L}_T^{\theta_4}(B_{p,r}^s)})$$

whenever $s > 0, 1 \leq p \leq \infty, 1 \leq \theta, \theta_1, \theta_2, \theta_3, \theta_4 \leq \infty$ and

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} = \frac{1}{\theta_3} + \frac{1}{\theta_4}.$$

As a direct corollary, one has

$$\|fg\|_{\tilde{L}_T^\theta(B_{p,r}^s)} \leq C\|f\|_{\tilde{L}_T^{\theta_1}(B_{p,r}^s)} \|g\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)}$$

whenever $s \geq d/p, \frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$.

In the next symmetrization, we meet with some composition functions. The following continuity result for compositions is used to estimate them.

Proposition 2.2. ([1]) Let $s > 0, 1 \leq p, r, \theta \leq \infty, F' \in W_{loc}^{[s]+1, \infty}(I; \mathbb{R})$ with $F(0) = 0, T \in (0, \infty]$ and $v \in \tilde{L}_T^\theta(B_{p,r}^s) \cap L_T^\infty(L^\infty)$. Then

$$\|F(f)\|_{\tilde{L}_T^\theta(B_{p,r}^s)} \leq C(1 + \|f\|_{L_T^\infty(L^\infty)})^{[s]+1} \|F'\|_{W^{[s]+1, \infty}} \|f\|_{\tilde{L}_T^\theta(B_{p,r}^s)}.$$

In addition, we present some estimates of commutators in homogeneous and inhomogeneous Chemin-Lerner spaces to bound commutators.

Proposition 2.3. *Let $1 < p < \infty$ and $1 \leq \rho \leq \infty$. Then there exists a generic constant $C > 0$ depending only on s, N such that*

$$\begin{cases} \| [f, \Delta_q] \mathcal{A} g \|_{L_T^\theta(L^p)} \leq C c_q 2^{-qs} \| \nabla f \|_{\tilde{L}_T^{\theta_1}(B_{p,1}^{s-1})} \| g \|_{\tilde{L}_T^{\theta_2}(B_{p,1}^s)}, & s = 1 + \frac{N}{p}, \\ \| [f, \Delta_q] g \|_{L_T^\theta(L^p)} \leq C c_q 2^{-q(s+1)} \| f \|_{\tilde{L}_T^{\theta_1}(\dot{B}_{p,1}^{\frac{N}{p}+1})} \| g \|_{\tilde{L}_T^{\theta_2}(\dot{B}_{p,1}^s)}, & s \in (-\frac{N}{p} - 1, \frac{N}{p}], \end{cases}$$

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$, and the operator $\mathcal{A} := \text{div}$ or ∇ . $\{c_q\}$ denotes a sequence such that $\|(c_q)\|_{l^1} \leq 1$, $\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2}$.

Finally, let us point out that all the properties described in the this section remain true in the periodic setting, see [5].

3 Local existence and blow-up criterion

It is convenient to obtain the main results, we first reformulate the compressible Euler-Maxwell system (1.1). Set

$$\begin{cases} \varrho_\pm(t, x) = \frac{2}{\gamma-1} \{ [n_\pm(\frac{t}{\sqrt{\gamma}}, x)]^{\frac{\gamma-1}{2}} - 1 \}, & v_\pm(t, x) = \frac{1}{\sqrt{\gamma}} u_\pm(\frac{t}{\sqrt{\gamma}}, x), \\ \tilde{E}(t, x) = \frac{1}{\sqrt{\gamma}} E(\frac{t}{\sqrt{\gamma}}, x), & \tilde{B}(t, x) = \frac{1}{\sqrt{\gamma}} B(\frac{t}{\sqrt{\gamma}}, x) - \bar{B}. \end{cases} \quad (3.1)$$

Then the system (1.1) can be reformulated, for classical solution $W = (\varrho_\pm, v_\pm, \tilde{E}, \tilde{B})$, as

$$\begin{cases} \partial_t \varrho_\pm + v_\pm \cdot \nabla \varrho_\pm + (\frac{\gamma-1}{2} \varrho_\pm + 1) \nabla \cdot v_\pm = 0, \\ \partial_t v_\pm + (\frac{\gamma-1}{2} \varrho_\pm + 1) \nabla \varrho_\pm + v_\pm \cdot \nabla v_\pm = \mp (\frac{1}{\sqrt{\gamma}} \tilde{E} + v_\pm \times (\tilde{B} + \bar{B})) - \frac{1}{\sqrt{\gamma}} v_\pm, \\ \partial_t \tilde{E} - \frac{1}{\sqrt{\gamma}} \nabla \times \tilde{B} = \frac{1}{\sqrt{\gamma}} v_+ + \frac{1}{\sqrt{\gamma}} [\Phi(\varrho_+) + \varrho_+] v_+ - \frac{1}{\sqrt{\gamma}} v_- - \frac{1}{\sqrt{\gamma}} [\Phi(\varrho_-) + \varrho_-] v_-, \\ \partial_t \tilde{B} + \frac{1}{\sqrt{\gamma}} \nabla \times \tilde{E} = 0, \\ \nabla \cdot \tilde{E} = -\frac{1}{\sqrt{\gamma}} [\Phi(\varrho_+) + \varrho_+] + \frac{1}{\sqrt{\gamma}} [\Phi(\varrho_-) + \varrho_-], \quad \nabla \cdot \tilde{B} = 0 \end{cases} \quad (3.2)$$

with the initial data

$$W|_{t=0} = W_0 := (\varrho_{\pm 0}, v_{\pm 0}, \tilde{E}_0, \tilde{B}_0) \quad (3.3)$$

satisfying the corresponding compatible conditions

$$\begin{cases} \nabla \cdot \tilde{E}_0 = -\frac{1}{\sqrt{\gamma}} [\Phi(\varrho_{+0}) + \varrho_{+0}] + \frac{1}{\sqrt{\gamma}} [\Phi(\varrho_{-0}) + \varrho_{-0}], \\ \nabla \cdot \tilde{B}_0 = 0. \end{cases} \quad (3.4)$$

Here the nonlinear function $\Phi(\cdot)$ in (3.2) is defined by

$$\Phi(\rho) = (\frac{\gamma-1}{2} \rho + 1)^{\frac{2}{\gamma-1}} - \rho - 1.$$

Notice that $\Phi(\rho)$ is a smooth function on the domain $\{\rho | \frac{\gamma-1}{2} \rho + 1 > 0\}$ satisfying $\Phi(0) = \Phi'(0) = 0$, which is a little different from that in [23].

For this reformulation, we have

Remark 3.1. The variable change is from the open set $\{(n_+, u_+, n_-, u_-, E, B) \in (0, +\infty) \times \mathbb{R}^N \times (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N\}$ to the open set $\{W \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \mid \frac{\gamma-1}{2} \varrho_{\pm} + 1 > 0\}$. It is easy to show that for classical solutions (n_{\pm}, u_{\pm}, E, B) away from vacuum, (1.1)-(1.2) is equivalent to (3.2)-(3.3) with $\frac{\gamma-1}{2} \varrho_{\pm} + 1 > 0$.

The simpler case of $\gamma = 1$ can be treated in the similar way by using the reformulation in terms of the enthalpy variable, see, *e.g.*, [23]. Without loss of generality, we focus on the system (3.2)-(3.3).

Next, let us write (3.2) as a symmetric hyperbolic system. Set

$$W_I = (\varrho_+, v_+, \varrho_-, v_-)^{\top}, \quad W_{II} = (\tilde{E}, \tilde{B})^{\top}, \quad W = (W_I, W_{II})^{\top}.$$

Then (3.2) is reduced to

$$\partial_t W + \sum_{j=1}^N A_j(W_I) \partial_{x_j} W = L(W), \quad (3.5)$$

where

$$A_j(W_I) = \begin{pmatrix} A_j^I(W_I) & 0 \\ 0 & A_j^{II} \end{pmatrix},$$

$$L(W) = \begin{pmatrix} 0 \\ -(\frac{1}{\sqrt{\gamma}} \tilde{E} + v_+ \times (\tilde{B} + \bar{B})) - \frac{1}{\sqrt{\gamma}} v_+ \\ 0 \\ (\frac{1}{\sqrt{\gamma}} \tilde{E} + v_- \times (\tilde{B} + \bar{B})) - \frac{1}{\sqrt{\gamma}} v_- \\ \frac{1}{\sqrt{\gamma}} v_+ + \frac{1}{\sqrt{\gamma}} [\Phi(\varrho_+) + \varrho_+] v_+ - \frac{1}{\sqrt{\gamma}} v_- - \frac{1}{\sqrt{\gamma}} [\Phi(\varrho_-) + \varrho_-] v_- \\ 0 \end{pmatrix}$$

with

$$A_j^I(W_I) = \begin{pmatrix} v_+^j & (\frac{\gamma-1}{2} \varrho_+ + 1) e_j^{\top} & 0 & 0 \\ (\frac{\gamma-1}{2} \varrho_+ + 1) e_j & v_+^j I_N & 0 & 0 \\ 0 & 0 & v_-^j & (\frac{\gamma-1}{2} \varrho_- + 1) e_j^{\top} \\ 0 & 0 & (\frac{\gamma-1}{2} \varrho_- + 1) e_j & v_-^j I_N \end{pmatrix},$$

$$A_j^{II} = \begin{pmatrix} 0 & P_j \\ P_j^{\top} & 0 \end{pmatrix}$$

and

$$P_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{\gamma}} \\ 0 & -\frac{1}{\sqrt{\gamma}} & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & -\frac{1}{\sqrt{\gamma}} \\ 0 & 0 & 0 \\ \frac{1}{\sqrt{\gamma}} & 0 & 0 \end{pmatrix}, \quad P_3 = \begin{pmatrix} 0 & \frac{1}{\sqrt{\gamma}} & 0 \\ -\frac{1}{\sqrt{\gamma}} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Here I_N denotes the unit matrix of order N and e_j is the N -dimensional vector where the j th component is one, others are zero. From the explicit structure of the block matrix $A_j(W_I)$

above, we see that (3.2) is a symmetric hyperbolic system on $G = \{W | \frac{\gamma-1}{2}\varrho_{\pm} + 1 > 0\}$ in the sense of Friedrichs. Based on the recent work [24] for generally symmetric hyperbolic systems, we get the local existence and uniqueness of classical solutions $W = (\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})$, which reads as follows.

Proposition 3.1. *Assume that $W_0 \in B_{2,1}^{s_c}$ satisfying $\frac{\gamma-1}{2}\varrho_{\pm 0} + 1 > 0$ and (3.4), then there exists a time $T_0 > 0$ (depending only on the initial data) such that*

(i) *Existence: the system (3.2)-(3.3) has a unique solution $W \in \mathcal{C}^1([0, T_0] \times \mathbb{R}^N)$ with $\frac{\gamma-1}{2}\varrho_{\pm} + 1 > 0$ for all $t \in [0, T_0]$ and $W \in \tilde{\mathcal{C}}_{T_0}(B_{2,1}^{s_c}) \cap \tilde{\mathcal{C}}_{T_0}^1(B_{2,1}^{s_c-1})$;*

(ii) *Blow-up criterion: if the maximal time $T^*(> T_0)$ of existence of such a solution is finite, then*

$$\limsup_{t \rightarrow T^*} \|W(t)\|_{B_{2,1}^{s_c}} = \infty$$

if and only if

$$\int_0^{T^*} \|\nabla W(t)\|_{L^\infty} dt = \infty.$$

Proof. From [24], it suffices to establish the blow-up criterion. We consider the symmetric system (3.5) with $L(W) \equiv 0$ for simplicity, since it is only responsible for the global well-posedness and large time behavior of solutions.

Applying the homogeneous operator $\dot{\Delta}_q$ to (3.5), we infer that $\dot{\Delta}_q W$ satisfies

$$\partial_t \dot{\Delta}_q W + \sum_{j=1}^N A_j(W_I) \dot{\Delta}_q \partial_{x_j} W = - \sum_{j=1}^N [\dot{\Delta}_q, A_j(W_I)] W_{x_j}, \quad (3.6)$$

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] := fg - gf$.

Perform the inner product with $\dot{\Delta}_q W$ on both sides of the equation (3.6) to get

$$\begin{aligned} & \langle \dot{\Delta}_q W, \dot{\Delta}_q W \rangle_t + \sum_{j=1}^N \langle A_j(W_I) \dot{\Delta}_q W, \dot{\Delta}_q W \rangle_{x_j} \\ &= -2 \sum_{j=1}^N \langle [\dot{\Delta}_q, A_j(W_I)] W_{x_j}, \dot{\Delta}_q W \rangle + \langle A_j(W_I)_{x_j} \dot{\Delta}_q W, \dot{\Delta}_q W \rangle. \end{aligned} \quad (3.7)$$

By integrating (3.7) with respect to x over \mathbb{R}^N , we get

$$\frac{d}{dt} \|\dot{\Delta}_q W\|_{L^2}^2 \lesssim \|[\dot{\Delta}_q, A_j(W_I)] W_{x_j}\|_{L^2} \|\dot{\Delta}_q W\|_{L^2} + \|A_j(W_I)_{x_j}\|_{L^\infty} \|\dot{\Delta}_q W\|_{L^2}^2. \quad (3.8)$$

Let $\epsilon > 0$ be a small number. Dividing (3.8) by $(\|\dot{\Delta}_q W\|_{L^2}^2 + \epsilon)^{1/2}$ gives

$$\begin{aligned} & \frac{d}{dt} \left(\|\dot{\Delta}_q W\|_{L^2}^2 + \epsilon \right)^{1/2} \\ & \lesssim \|[\dot{\Delta}_q, A_j(W_I)] W_{x_j}\|_{L^2} + \|A_j(W_I)_{x_j}\|_{L^\infty} \|\dot{\Delta}_q W\|_{L^2} \\ & \lesssim c_q(t) 2^{-qs_c} (\|A_j(W_I)_{x_j}\|_{L^\infty} \|W\|_{\dot{B}_{2,1}^{s_c}} + \|A_j(W_I)_{x_j}\|_{\dot{B}_{2,1}^{s_c}} \|\nabla W\|_{L^\infty}) \\ & \quad + \|A_j(W_I)_{x_j}\|_{L^\infty} \|\dot{\Delta}_q W\|_{L^2}, \end{aligned} \quad (3.9)$$

where we used the stationary cases of estimates of commutator in [2](Lemma 2.100, P.112) and the sequence $\{c_q(t)\}$ satisfying $\|c_q(t)\|_{\ell^1} \leq 1$, for all $t \in [0, T_0]$.

Taking a time integration and passing to the limit $\epsilon \rightarrow 0$, we arrive at

$$\|W(t)\|_{\dot{B}_{2,1}^{s_c}} \lesssim \|W_0\|_{\dot{B}_{2,1}^{s_c}} + \int_0^{T_0} \|\nabla W(\tau)\|_{L^\infty} \|W(\tau)\|_{\dot{B}_{2,1}^{s_c}} d\tau, \quad (3.10)$$

since we note that the fact $W(t, x) \in \mathcal{O}_1$ (a bounded open convex set in \mathbb{R}^{4N+2}) for any $(t, x) \in [0, T_0] \times \mathbb{R}^N$, see [24].

On the other hand, we take the L^2 -inner product on (3.5) with W . It is not difficult to obtain

$$\|W(t)\|_{L^2} \lesssim \|W_0\|_{L^2} + \int_0^{T_0} \|\nabla W(\tau)\|_{L^\infty} \|W(\tau)\|_{L^2} d\tau. \quad (3.11)$$

Adding (3.10) to (3.11), from (1) in Lemma 2.2, we have

$$\|W(t)\|_{B_{2,1}^{s_c}} \lesssim \|W_0\|_{B_{2,1}^{s_c}} + \int_0^{T_0} \|\nabla W(\tau)\|_{L^\infty} \|W(\tau)\|_{B_{2,1}^{s_c}} d\tau. \quad (3.12)$$

Gronwall's inequality implies

$$\sup_{t \in [0, T_0]} \|W(t)\|_{B_{2,1}^{s_c}} \lesssim \|W_0\|_{B_{2,1}^{s_c}} \exp \left(\int_0^{T_0} \|\nabla W(\tau)\|_{L^\infty} d\tau \right). \quad (3.13)$$

Besides, we have the following obvious inequalities

$$\int_0^{T_0} \|\nabla W(\tau)\|_{L^\infty} d\tau \lesssim \int_0^{T_0} \|W(\tau)\|_{B_{2,1}^{s_c}} d\tau \lesssim T_0 \sup_{t \in [0, T_0]} \|W(t)\|_{B_{2,1}^{s_c}}. \quad (3.14)$$

Hence the blow-up criterion follows (3.13) and (3.14) immediately. This completes Proposition 3.1. \square

4 Global well-posedness

In this section, we focus on the global existence of classical solutions to (3.2)-(3.3). For that purpose, we first derive a crucial *a priori* estimate in the whole space, which is comprised in the following proposition.

Proposition 4.1. *There exist some positive constants δ_1, μ_1 and C_1 such that for any $T > 0$, if*

$$\|(\varrho_\pm, v_\pm, \tilde{E}, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \leq \delta_1, \quad (4.1)$$

then

$$\begin{aligned} & \|(\varrho_\pm, v_\pm, \tilde{E}, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \\ & + \mu_1 \left\{ \|(\varrho_+ - \varrho_-, v_\pm)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|(\nabla \varrho_\pm, \tilde{E})\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|\nabla \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} \right\} \\ & \leq C_1 \|(\varrho_{\pm 0}, v_{\pm 0}, \tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}}. \end{aligned} \quad (4.2)$$

Actually, the proof of Proposition 4.1 is to capture the dissipation rates from contributions of $(\varrho_\pm, v_\pm, \tilde{E}, \tilde{B})$ in turn by using the low- and high-frequency decomposition methods. For clarity, we divide it into several lemmas.

Lemma 4.1. *If $W \in \tilde{\mathcal{C}}_T(B_{2,1}^{sc}) \cap \tilde{\mathcal{C}}_T^1(B_{2,1}^{sc-1})$ is a solution of (3.2)-(3.3) for any $T > 0$, then the following estimate holds:*

$$\begin{aligned} & \|W\|_{\tilde{L}_T^\infty(B_{2,1}^{sc})} + \mu_2 \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{sc})} \\ & \lesssim \|W_0\|_{B_{2,1}^{sc}} + \sqrt{\|W\|_{\tilde{L}_T^\infty(B_{2,1}^{sc})}} \left(\|(\nabla \varrho_+, \nabla \varrho_-, \tilde{E})\|_{\tilde{L}_T^2(B_{2,1}^{sc-1})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{sc})} \right), \end{aligned} \quad (4.3)$$

where μ_2 is a positive constant.

Proof. The proof is divided into three steps.

Step 1. The $\tilde{L}_T^2(B_{2,1}^{sc})$ estimates of (v_+, v_-) .

Indeed, applying the homogeneous localization operator $\dot{\Delta}_q (q \in \mathbb{Z})$ to (3.2), we infer that

$$\left\{ \begin{aligned} & \partial_t \dot{\Delta}_q \varrho_+ + \dot{\Delta}_q \operatorname{div} v_+ \\ & \quad = -(\varrho_+ \cdot \nabla) \dot{\Delta}_q \varrho_+ + [v_+, \dot{\Delta}_q] \cdot \nabla \varrho_+ - \frac{\gamma-1}{2} ([\dot{\Delta}_q, \varrho_+] \operatorname{div} v_+ + \varrho_+ \dot{\Delta}_q \operatorname{div} v_+), \\ & \partial_t \dot{\Delta}_q \varrho_- + \dot{\Delta}_q \operatorname{div} v_- \\ & \quad = -(\varrho_- \cdot \nabla) \dot{\Delta}_q \varrho_- + [v_-, \dot{\Delta}_q] \cdot \nabla \varrho_- - \frac{\gamma-1}{2} ([\dot{\Delta}_q, \varrho_-] \operatorname{div} v_- + \varrho_- \dot{\Delta}_q \operatorname{div} v_-), \\ & \partial_t \dot{\Delta}_q v_+ + \dot{\Delta}_q \nabla \varrho_+ + \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q v_+ + \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q \tilde{E} + \dot{\Delta}_q v_+ \times \tilde{B} \\ & \quad = -(\varrho_+ \cdot \nabla) \dot{\Delta}_q v_+ + [v_+, \dot{\Delta}_q] \cdot \nabla v_+ - \frac{\gamma-1}{2} ([\dot{\Delta}_q, \varrho_+] \nabla \varrho_+ + \varrho_+ \dot{\Delta}_q \nabla \varrho_+) - \dot{\Delta}_q (v_+ \times \tilde{B}), \\ & \partial_t \dot{\Delta}_q v_- + \dot{\Delta}_q \nabla \varrho_- + \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q v_- - \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q \tilde{E} - \dot{\Delta}_q v_- \times \tilde{B} \\ & \quad = -(\varrho_- \cdot \nabla) \dot{\Delta}_q v_- + [v_-, \dot{\Delta}_q] \cdot \nabla v_- - \frac{\gamma-1}{2} ([\dot{\Delta}_q, \varrho_-] \nabla \varrho_- + \varrho_- \dot{\Delta}_q \nabla \varrho_-) + \dot{\Delta}_q (v_- \times \tilde{B}), \\ & \partial_t \dot{\Delta}_q \tilde{E} - \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q \nabla \times \tilde{B} \\ & \quad = \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q v_+ + \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q [(\Phi(\varrho_+) + \varrho_+) v_+] - \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q v_- - \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q [(\Phi(\varrho_-) + \varrho_-) v_-], \\ & \partial_t \dot{\Delta}_q \tilde{B} + \frac{1}{\sqrt{\gamma}} \dot{\Delta}_q \nabla \times \tilde{E} = 0, \end{aligned} \right. \quad (4.4)$$

where the commutator $[\cdot, \cdot]$ is defined by $[f, g] = fg - gf$.

Then multiplying the first two equations of (4.4) by $\dot{\Delta}_q \varrho_+$, $\dot{\Delta}_q \varrho_-$, the third one by $\dot{\Delta}_q v_+$, the fourth one by $\dot{\Delta}_q v_-$, respectively, and adding the resulting equations together after integrating them over \mathbb{R}^N , we get

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_q(\varrho_+, \varrho_-, v_+, v_-)\|_{L^2}^2 + \frac{1}{\sqrt{\gamma}} \|\dot{\Delta}_q(v_+, v_-)\|_{L^2}^2 + \frac{1}{\sqrt{\gamma}} \langle \dot{\Delta}_q \tilde{E}, \dot{\Delta}_q v_+ \rangle - \frac{1}{\sqrt{\gamma}} \langle \dot{\Delta}_q \tilde{E}, \dot{\Delta}_q v_- \rangle \\ & \quad = \sum_{i=+,-} I_1^i(t) + \sum_{i=+,-} I_2^i(t), \end{aligned} \quad (4.5)$$

where we have used the facts $(\dot{\Delta}_q v_\pm \times \tilde{B}) \cdot \dot{\Delta}_q v_\pm = 0$. The energy functions in the right-side of (4.5) are defined by

$$\begin{aligned} I_1^+(t) & := \frac{1}{2} \langle \operatorname{div} v_+, (|\dot{\Delta}_q \varrho_+|^2 + |\dot{\Delta}_q v_+|^2) \rangle + \frac{\gamma-1}{2} \langle \dot{\Delta}_q \varrho_+, \nabla \varrho_+ \cdot \dot{\Delta}_q v_+ \rangle \\ & \quad - \langle \dot{\Delta}_q (v_+ \times \tilde{B}), \dot{\Delta}_q v_+ \rangle, \end{aligned}$$

$$I_1^-(t) := \frac{1}{2} \langle \operatorname{div} v_-, (|\dot{\Delta}_q \varrho_-|^2 + |\dot{\Delta}_q v_-|^2) \rangle + \frac{\gamma-1}{2} \langle \dot{\Delta}_q \varrho_-, \nabla \varrho_- \cdot \dot{\Delta}_q v_- \rangle \\ + \langle \dot{\Delta}_q (v_- \times \tilde{B}), \dot{\Delta}_q v_- \rangle,$$

and

$$I_2^+(t) := \langle [v_+, \dot{\Delta}_q] \cdot \nabla \varrho_+, \dot{\Delta}_q \varrho_+ \rangle + \langle [v_+, \dot{\Delta}_q] \cdot \nabla v_+, \dot{\Delta}_q v_+ \rangle \\ - \frac{\gamma-1}{2} \langle [\dot{\Delta}_q, \varrho_+] \operatorname{div} v_+, \dot{\Delta}_q \varrho_+ \rangle - \frac{\gamma-1}{2} \langle [\dot{\Delta}_q, \varrho_+] \nabla \varrho_+, \dot{\Delta}_q v_+ \rangle, \\ I_2^-(t) := \langle [v_-, \dot{\Delta}_q] \cdot \nabla \varrho_-, \dot{\Delta}_q \varrho_- \rangle + \langle [v_-, \dot{\Delta}_q] \cdot \nabla v_-, \dot{\Delta}_q v_- \rangle \\ - \frac{\gamma-1}{2} \langle [\dot{\Delta}_q, \varrho_-] \operatorname{div} v_-, \dot{\Delta}_q \varrho_- \rangle - \frac{\gamma-1}{2} \langle [\dot{\Delta}_q, \varrho_-] \nabla \varrho_-, \dot{\Delta}_q v_- \rangle.$$

On the other hand, multiplying the fifth equation of (4.4) by $\dot{\Delta}_q \tilde{E}$ and the sixth one by $\dot{\Delta}_q \tilde{B}$, and adding the resulting equations together after integrating them over \mathbb{R}^N implies

$$\frac{1}{2} \frac{d}{dt} \|\dot{\Delta}_q (\tilde{E}, \tilde{B})\|_{L^2}^2 - \frac{1}{\sqrt{\gamma}} \langle \dot{\Delta}_q \tilde{E}, \dot{\Delta}_q v_+ \rangle + \frac{1}{\sqrt{\gamma}} \langle \dot{\Delta}_q \tilde{E}, \dot{\Delta}_q v_- \rangle \\ = \sum_{i=+,-} I_3^i(t), \quad (4.6)$$

where we used the vector formula $\nabla \cdot (\vec{f} \times \vec{g}) = (\nabla \times \vec{f}) \cdot \vec{g} - (\nabla \times \vec{g}) \cdot \vec{f}$ and $I_3^+(t), I_3^-(t)$ are given by

$$I_3^+(t) := \frac{1}{\sqrt{\gamma}} \langle \dot{\Delta}_q [(\Phi(\varrho_+) + \varrho_+) v_+], \dot{\Delta}_q \tilde{E} \rangle, \\ I_3^-(t) := -\frac{1}{\sqrt{\gamma}} \langle \dot{\Delta}_q [(\Phi(\varrho_-) + \varrho_-) v_-], \dot{\Delta}_q \tilde{E} \rangle.$$

In what follows, we begin to bound these nonlinear terms. Firstly, with the aid of Cauchy-Schwartz inequality, we have

$$\int_0^T |I_1^+(t)| dt \\ \lesssim \|(\nabla \varrho_+, \nabla v_+)\|_{L_T^2(L^\infty)} \left(2^{-q} \|\dot{\Delta}_q \nabla \varrho_+\|_{L_T^2(L^2)} \|\dot{\Delta}_q \varrho_+\|_{L_T^\infty(L^2)} \right. \\ \left. + 2^{-q} \|\dot{\Delta}_q \nabla \varrho_+\|_{L_T^2(L^2)} \|\dot{\Delta}_q v_+\|_{L_T^\infty(L^2)} + \|\dot{\Delta}_q v_+\|_{L_T^2(L^2)} \|\dot{\Delta}_q v_+\|_{L_T^\infty(L^2)} \right) \\ + \|\dot{\Delta}_q (v_+ \times \tilde{B})\|_{L_T^2(L^2)} \|\dot{\Delta}_q v_+\|_{L_T^2(L^2)}, \quad (4.7)$$

furthermore, multiplying the factor 2^{2qsc} on both sides of (4.7) gives

$$2^{2qsc} \int_0^T |I_1^+(t)| dt \\ \lesssim c_q^2 \|(\varrho_+, v_+)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{sc})} \left(\|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{sc-1})}^2 + \|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{sc-1})} \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{sc})} \right. \\ \left. + \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{sc})}^2 \right) + c_q^2 \|v_+ \times \tilde{B}\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{sc})} \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{sc})}, \quad (4.8)$$

where we used the embedding properties in Lemma 2.2 and Remark 2.1. Here and below, $\{c_q\}$ denotes some sequence which satisfies $\|(c_q)\|_{l^1} \leq 1$ although each $\{c_q\}$ is possibly different in (4.8). Similarly, we have

$$\begin{aligned}
& 2^{2qs_c} \int_0^T |I_1^-(t)| dt \\
& \lesssim c_q^2 \|(\varrho_-, v_-)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \left(\|\nabla \varrho_-\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c-1})}^2 + \|\nabla \varrho_-\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c-1})} \|v_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \right. \\
& \quad \left. + \|v_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})}^2 \right) + c_q^2 \|v_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \times \tilde{B} \|\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})\| v_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})}. \tag{4.9}
\end{aligned}$$

Secondly, we turn to estimate the commutators occurring in $I_2^\pm(t)$. Indeed, we arrive at

$$\begin{aligned}
& 2^{2qs_c} \int_0^T |I_2^+(t)| dt \\
& \lesssim 2^{qs_c} \left(\|[\varrho_+, \dot{\Delta}_q] \cdot \nabla \varrho_+\|_{L_T^2(L^2)} + \|[\varrho_+, \dot{\Delta}_q] \operatorname{div} v_+\|_{L_T^2(L^2)} \right) 2^{q(s_c-1)} \|\dot{\Delta}_q \nabla \varrho_+\|_{L_T^2(L^2)} \\
& \quad + 2^{qs_c} \left(\|[\varrho_+, \dot{\Delta}_q] \cdot \nabla v_+\|_{L_T^2(L^2)} + \|[\varrho_+, \dot{\Delta}_q] \nabla \varrho_+\|_{L_T^2(L^2)} \right) 2^{qs_c} \|\dot{\Delta}_q v_+\|_{L_T^2(L^2)}. \tag{4.10}
\end{aligned}$$

Taking advantage of the estimates of commutator in Proposition 2.3, we obtain

$$\begin{aligned}
& 2^{2qs_c} \int_0^T |I_2^+(t)| dt \\
& \lesssim c_q^2 \left(\|v_+\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} + \|\varrho_+\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \|\nabla v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \right) \|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \\
& \quad + c_q^2 \left(\|v_+\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \|\nabla v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} + \|\varrho_+\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \right) \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \\
& \lesssim c_q^2 \|(\varrho_+, v_+)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \\
& \quad \times \left(\|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})}^2 + \|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} + \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})}^2 \right). \tag{4.11}
\end{aligned}$$

Similarly,

$$\begin{aligned}
& 2^{2qs_c} \int_0^T |I_2^-(t)| dt \\
& \lesssim c_q^2 \|(\varrho_-, v_-)\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \\
& \quad \times \left(\|\nabla \varrho_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})}^2 + \|\nabla \varrho_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \|v_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} + \|v_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})}^2 \right). \tag{4.12}
\end{aligned}$$

Thirdly, the composition functions $I_3^\pm(t)$ can be estimated as

$$\begin{aligned}
& 2^{2qs_c} \int_0^T |I_3^+(t)| dt \\
& \lesssim c_q^2 \|\tilde{E}\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \left(\|\Phi(\varrho_+) v_+\|_{L_T^1(\dot{B}_{2,1}^{s_c})} + \|\varrho_+ v_+\|_{L_T^1(\dot{B}_{2,1}^{s_c})} \right) \\
& \lesssim c_q^2 \|\tilde{E}\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \left(\|\Phi(\varrho_+)\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} + \|\varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \right) \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \\
& \lesssim c_q^2 \|\tilde{E}\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \|\nabla \varrho_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \|v_+\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})}, \tag{4.13}
\end{aligned}$$

where we used the corresponding homogeneous cases of Propositions 2.1-2.2. Similarly,

$$2^{2qs_c} \int_0^T |I_3^-(t)| dt \lesssim c_q^2 \|\tilde{E}\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} \|\nabla \varrho_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \|v_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})}. \quad (4.14)$$

Together with the equalities (4.5)-(4.6) and inequalities (4.8)-(4.9), (4.11)-(4.14), we conclude that there exists a constant $\tilde{\mu}_2 > 0$ such that

$$\begin{aligned} & 2^{2qs_c} \|\dot{\Delta}_q W(t)\|_{L^2}^2 + \tilde{\mu}_2^2 2^{2qs_c} \|(\dot{\Delta}_q v_+, \dot{\Delta}_q v_-)\|_{L_t^2(L^2)}^2 \\ & \lesssim 2^{2qs_c} \|\dot{\Delta}_q W_0\|_{L^2}^2 + \text{the right sides of } \left\{ (4.8) - (4.9), (4.11) - (4.14) \right\}. \end{aligned} \quad (4.15)$$

Then it follows from the classical Young's inequality ($\sqrt{fg} \leq (f+g)/2$, $f, g \geq 0$) that

$$\begin{aligned} & 2^{qs_c} \|\dot{\Delta}_q W\|_{L_T^\infty(L^2)} + \tilde{\mu}_2 2^{qs_c} \|(\dot{\Delta}_q v_+, \dot{\Delta}_q v_-)\|_{L_T^2(L^2)} \\ & \lesssim 2^{qs_c} \|\dot{\Delta}_q W_0\|_{L^2} + c_q \sqrt{\|W\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})}} \left(\|\nabla \varrho_\pm\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} + \|v_\pm\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \right). \end{aligned} \quad (4.16)$$

Hence, summing up (4.16) on $q \in \mathbb{Z}$ gives immediately

$$\begin{aligned} & \|W\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})} + \tilde{\mu}_2 \|(v_+, v_-)\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \\ & \lesssim \|W_0\|_{\dot{B}_{2,1}^{s_c}} + \sqrt{\|W\|_{\tilde{L}_T^\infty(\dot{B}_{2,1}^{s_c})}} \left(\|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \right). \end{aligned} \quad (4.17)$$

Step 2. The $L_T^2(L^2)$ estimates of (v_+, v_-) .

It follows from (3.2) and usual energy methods, we can get the equalities

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\varrho_+, \varrho_-, v_+, v_-)\|_{L^2}^2 + \frac{1}{\sqrt{\gamma}} \|(v_+, v_-)\|_{L^2}^2 + \frac{1}{\sqrt{\gamma}} \langle \tilde{E}, v_+ \rangle - \frac{1}{\sqrt{\gamma}} \langle \tilde{E}, v_- \rangle \\ & = \frac{\gamma-1}{2} \langle \varrho_+, v_+ \cdot \nabla \varrho_+ \rangle + \frac{\gamma-1}{2} \langle \varrho_-, v_- \cdot \nabla \varrho_- \rangle - \langle \nabla v_+, v_+^2 \rangle - \langle \nabla v_-, v_-^2 \rangle \\ & \quad - \langle v_+ \times \tilde{B}, v_+ \rangle + \langle v_- \times \tilde{B}, v_- \rangle, \end{aligned} \quad (4.18)$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \|(\tilde{E}, \tilde{B})\|_{L^2}^2 - \frac{1}{\sqrt{\gamma}} \langle \tilde{E}, v_+ \rangle + \frac{1}{\sqrt{\gamma}} \langle \tilde{E}, v_- \rangle \\ & = \frac{1}{\sqrt{\gamma}} \langle [(\Phi(\varrho_+) + \varrho_+)v_+], \tilde{E} \rangle - \frac{1}{\sqrt{\gamma}} \langle [(\Phi(\varrho_-) + \varrho_-)v_-], \tilde{E} \rangle. \end{aligned} \quad (4.19)$$

Combine (4.18) and (4.19) to get

$$\begin{aligned} & \frac{d}{dt} \|W\|_{L^2}^2 + \frac{2}{\sqrt{\gamma}} \|(v_+, v_-)\|_{L^2}^2 \\ & \lesssim \|(\varrho_+, \varrho_-)\|_{L^\infty} \left(\|v_+\|_{L^2} \|\nabla \varrho_+\|_{L^2} + \|v_-\|_{L^2} \|\nabla \varrho_-\|_{L^2} \right) \\ & \quad + \|(\nabla v_+, \nabla v_-, \tilde{B})\|_{L^\infty} \left(\|v_+\|_{L^2}^2 + \|v_-\|_{L^2}^2 \right) \\ & \quad + \|(\Phi(\varrho_+), \varrho_+, \Phi(\varrho_-), \varrho_-)\|_{L^\infty} \left(\|v_+\|_{L^2} + \|v_-\|_{L^2} \right) \|\tilde{E}\|_{L^2}. \end{aligned} \quad (4.20)$$

By integrating (4.20) with respect to $t \in [0, T]$, we obtain

$$\begin{aligned}
& \|W\|_{L_T^\infty(L^2)} + (2\gamma^{-\frac{1}{2}})^{\frac{1}{2}} \|(v_+, v_-)\|_{L_T^2(L^2)} \\
& \lesssim \|W_0\|_{L^2} + \sqrt{\|(\varrho_+, \varrho_-, \nabla v_+, \nabla v_-, \tilde{B}, \Phi(\varrho_+), \Phi(\varrho_-))\|_{L_T^\infty(L^\infty)} \\
& \quad \times \left(\|(\nabla \varrho_+, \nabla \varrho_-)\|_{L_T^2(L^2)} + \|(v_+, v_-, \tilde{E})\|_{L_T^2(L^2)} \right) \\
& \lesssim \|W_0\|_{L^2} + \sqrt{\|(\varrho_\pm, v_\pm, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \\
& \quad \times \left(\|(\nabla \varrho_+, \nabla \varrho_-)\|_{L_T^2(L^2)} + \|(v_+, v_-)\|_{L_T^2(L^2)} + \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \right)}. \tag{4.21}
\end{aligned}$$

Step 3. The $\tilde{L}_T^2(B_{2,1}^{s_c})$ estimates of (v_+, v_-) .

Recently, Xu and Kawashima [24] obtained an elementary fact that indicates the connection between the homogeneous Chemin-Lerner's spaces and inhomogeneous Chemin-Lerner's spaces. Precisely, it reads as follows: let $s > 0, 1 \leq \theta, p, r \leq +\infty$. When $\theta \geq r$, it holds that

$$L_T^\theta(L^p) \cap \tilde{L}_T^\theta(\dot{B}_{p,r}^s) = \tilde{L}_T^\theta(B_{p,r}^s) \tag{4.22}$$

for any $T > 0$. Notice this fact and (1) in Lemma 2.2, the inequality (4.3) directly follows from (4.17) and (4.21) with $\mu_2 = \min(\tilde{\mu}_2, (2\gamma^{-\frac{1}{2}})^{\frac{1}{2}})$. \square

Lemma 4.2. *If $W \in \tilde{C}_T(B_{2,1}^{s_c}) \cap \tilde{C}_T^1(B_{2,1}^{s_c-1})$ is a solution of (3.2)-(3.3) for any $T > 0$, then the following estimate holds:*

$$\begin{aligned}
& \|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\
& \lesssim (\|(\varrho_\pm, v_\pm)\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} + \|(\varrho_\pm 0, v_\pm 0)\|_{B_{2,1}^{s_c}} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \\
& \quad + \|(\varrho_\pm, v_\pm, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \left(\|(\nabla \varrho_\pm, \varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right). \tag{4.23}
\end{aligned}$$

Proof. The proof is divided into two claims for clarity.

Claim 1. Under the assumptions of Lemma 4.2, it holds that

$$\begin{aligned}
& \|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\
& \lesssim (\|(\varrho_\pm, v_\pm)\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} + \|(\varrho_\pm 0, v_\pm 0)\|_{B_{2,1}^{s_c}} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \\
& \quad + \|(\varrho_\pm, v_\pm, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \left(\|(\nabla \varrho_\pm, \varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right). \tag{4.24}
\end{aligned}$$

To do this, the first four equations of (3.2) can be rewritten as

$$\partial_t \varrho_+ + \operatorname{div} v_+ = f_1^+ \tag{4.25}$$

$$\partial_t \varrho_- + \operatorname{div} v_- = f_1^- \tag{4.26}$$

$$\partial_t v_+ + \nabla \varrho_+ + \frac{1}{\sqrt{\gamma}} v_+ + \frac{1}{\sqrt{\gamma}} \tilde{E} = f_2^+ \quad (4.27)$$

$$\partial_t v_- + \nabla \varrho_- + \frac{1}{\sqrt{\gamma}} v_- - \frac{1}{\sqrt{\gamma}} \tilde{E} = f_2^- \quad (4.28)$$

where

$$\begin{cases} f_1^+ = -v_+ \cdot \nabla \varrho_+ - \frac{\gamma-1}{2} \varrho_+ \operatorname{div} v_+, \\ f_1^- = -v_- \cdot \nabla \varrho_- - \frac{\gamma-1}{2} \varrho_- \operatorname{div} v_-, \\ f_2^+ = -v_+ \cdot \nabla v_+ - \frac{\gamma-1}{2} \varrho_+ \nabla \varrho_+ - v_+ \times (\tilde{B} + \bar{B}), \\ f_2^- = -v_- \cdot \nabla v_- - \frac{\gamma-1}{2} \varrho_- \nabla \varrho_- + v_- \times (\tilde{B} + \bar{B}). \end{cases}$$

Applying the *inhomogeneous* operator $\Delta_q (q \geq -1)$ to (4.27) and multiplying the resulting equality by $\Delta_q \nabla \varrho_+$ implies

$$\begin{aligned} & \frac{d}{dt} \langle \Delta_q v_+, \Delta_q \nabla \varrho_+ \rangle + \|\Delta_q \nabla \varrho_+\|_{L^2}^2 + \frac{1}{\gamma} \|\Delta_q \varrho_+\|_{L^2}^2 - \frac{1}{\gamma} \langle \Delta_q \varrho_-, \Delta_q \varrho_+ \rangle \\ = & -\frac{1}{\gamma} \langle \Delta_q \Phi(\varrho_+), \Delta_q \varrho_+ \rangle + \frac{1}{\gamma} \langle \Delta_q \Phi(\varrho_-), \Delta_q \varrho_+ \rangle + \langle \Delta_q f_2^+, \Delta_q \nabla \varrho_+ \rangle \\ & - \frac{1}{\sqrt{\gamma}} \langle \Delta_q v_+, \Delta_q \nabla \varrho_+ \rangle + \|\Delta_q \operatorname{div} v_+\|_{L^2}^2 - \langle \Delta_q f_1^+, \Delta_q \operatorname{div} v_+ \rangle, \end{aligned} \quad (4.29)$$

where we used the last equation of (3.2) and (4.25).

In a similar way as above, we have

$$\begin{aligned} & \frac{d}{dt} \langle \Delta_q v_-, \Delta_q \nabla \varrho_- \rangle + \|\Delta_q \nabla \varrho_-\|_{L^2}^2 + \frac{1}{\gamma} \|\Delta_q \varrho_-\|_{L^2}^2 - \frac{1}{\gamma} \langle \Delta_q \varrho_+, \Delta_q \varrho_- \rangle \\ = & -\frac{1}{\gamma} \langle \Delta_q \Phi(\varrho_-), \Delta_q \varrho_- \rangle + \frac{1}{\gamma} \langle \Delta_q \Phi(\varrho_+), \Delta_q \varrho_- \rangle + \langle \Delta_q f_2^-, \Delta_q \nabla \varrho_- \rangle \\ & - \frac{1}{\sqrt{\gamma}} \langle \Delta_q v_-, \Delta_q \nabla \varrho_- \rangle + \|\Delta_q \operatorname{div} v_-\|_{L^2}^2 - \langle \Delta_q f_1^-, \Delta_q \operatorname{div} v_- \rangle. \end{aligned} \quad (4.30)$$

Furthermore, we add (4.29) to (4.30) to get

$$\begin{aligned} & \frac{d}{dt} (\langle \Delta_q v_+, \Delta_q \nabla \varrho_+ \rangle + \langle \Delta_q v_-, \Delta_q \nabla \varrho_- \rangle) + \|(\Delta_q \nabla \varrho_+, \Delta_q \nabla \varrho_-)\|_{L^2}^2 + \frac{1}{\gamma} \|\Delta_q \varrho_+ - \Delta_q \varrho_-\|_{L^2}^2 \\ = & \|\Delta_q \operatorname{div} v_+\|_{L^2}^2 + \|\Delta_q \operatorname{div} v_-\|_{L^2}^2 + \frac{1}{\gamma} \langle \Delta_q \Phi(\varrho_+), \Delta_q \varrho_- - \Delta_q \varrho_+ \rangle + \langle \Delta_q f_2^+, \Delta_q \nabla \varrho_+ \rangle \\ & + \langle \Delta_q f_2^-, \Delta_q \nabla \varrho_- \rangle + \frac{1}{\gamma} \langle \Delta_q \Phi(\varrho_-), \Delta_q \varrho_+ - \Delta_q \varrho_- \rangle - \frac{1}{\sqrt{\gamma}} \langle \Delta_q v_+, \Delta_q \nabla \varrho_+ \rangle \\ & - \frac{1}{\sqrt{\gamma}} \langle \Delta_q v_-, \Delta_q \nabla \varrho_- \rangle - \langle \Delta_q f_1^+, \Delta_q \operatorname{div} v_+ \rangle - \langle \Delta_q f_1^-, \Delta_q \operatorname{div} v_- \rangle. \end{aligned} \quad (4.31)$$

From Young's inequality, there exists a constant $\mu_3 > 0$ such that

$$\begin{aligned} & \frac{d}{dt} (\langle \Delta_q v_+, \Delta_q \nabla \varrho_+ \rangle + \langle \Delta_q v_-, \Delta_q \nabla \varrho_- \rangle) + \mu_3^2 (\|(\Delta_q \nabla \varrho_+, \Delta_q \nabla \varrho_-)\|_{L^2}^2 + \|\Delta_q \varrho_+ - \Delta_q \varrho_-\|_{L^2}^2) \\ \lesssim & \|(\Delta_q \operatorname{div} v_+, \Delta_q \operatorname{div} v_-)\|_{L^2}^2 + \|(\Delta_q v_+, \Delta_q v_-)\|_{L^2}^2 + \|\Delta_q (\Phi(\varrho_+) - \Phi(\varrho_-))\|_{L^2}^2 \\ & + \left(\|\Delta_q f_1^+\|_{L^2}^2 + \|\Delta_q f_2^+\|_{L^2}^2 \right) + \left(\|\Delta_q f_1^-\|_{L^2}^2 + \|\Delta_q f_2^-\|_{L^2}^2 \right). \end{aligned} \quad (4.32)$$

By integrating (4.32) with respect to $t \in [0, T]$, and multiplying the factor $2^{2q(s_c-1)}$ on both sides of the resulting inequality, we obtain

$$\begin{aligned}
& \mu_3 2^{q(s_c-1)} (\|(\Delta_q \nabla \varrho_+, \Delta_q \nabla \varrho_-)\|_{L_T^2(L^2)} + \|\Delta_q \varrho_+ - \Delta_q \varrho_-\|_{L_T^2(L^2)}) \\
& \lesssim c_q (\|(\varrho_\pm, v_\pm)\|_{\tilde{L}_T^\infty(L^2)} + \|(\varrho_{\pm 0}, v_{\pm 0})\|_{B_{2,1}^{s_c}}) \\
& \quad + c_q \left(\|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|\Phi(\varrho_+) - \Phi(\varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \right. \\
& \quad \left. + \|f_1^+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|f_2^+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|f_1^-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|f_2^-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \right). \quad (4.33)
\end{aligned}$$

Now we estimate nonlinear terms in the right-side of (4.33) in turn. Firstly,

$$\begin{aligned}
\|f_1^+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} & \lesssim \|v_+ \cdot \nabla \varrho_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|\varrho_+ \operatorname{div} v_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\
& \lesssim \|v_+\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|\nabla \varrho_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|\varrho_+\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|v_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c})}, \quad (4.34)
\end{aligned}$$

where we used Proposition 2.1 and Remark 2.1. Similarly, we have

$$\|f_1^-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \lesssim \|v_-\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|\nabla \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|\varrho_-\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|v_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c})}; \quad (4.35)$$

$$\begin{aligned}
\|f_2^+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} & \lesssim \|v_+ \cdot \nabla v_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|\varrho_+ \nabla \varrho_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|v_+ \times (\tilde{B} + \bar{B})\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\
& \lesssim \|v_+\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|v_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|\varrho_+\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|\nabla \varrho_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\
& \quad + (1 + \|\tilde{B}\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}) \|v_+\|_{\tilde{L}_T^2(B_{2,1}^{s_c})}; \quad (4.36)
\end{aligned}$$

$$\begin{aligned}
\|f_2^-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} & \lesssim \|v_-\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|v_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|\varrho_-\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|\nabla \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\
& \quad + (1 + \|\tilde{B}\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}) \|v_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c})}. \quad (4.37)
\end{aligned}$$

To estimate the continuity of compositions $\Phi(\varrho_+) - \Phi(\varrho_-)$, we need the further estimates rather than that in Proposition 2.2. Indeed, the Proposition 5.1 in Appendix 5 will be used, which is a natural generalization about the corresponding stationary case in [2]. In addition, we recall that $\Phi(\rho)$ is a smooth function on the domain $\{\rho | \frac{\gamma-1}{2}\rho + 1 > 0\}$ satisfying $\Phi'(0) = 0$. Hence, take $s = s_c - 1, \theta = 2, \theta_1 = \theta_4 = 2, \theta_2 = \theta_3 = \infty, p = 2, r = 1$ in (4.56) to get

$$\begin{aligned}
& \|\Phi(\varrho_+) - \Phi(\varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\
& \lesssim \|\varrho_+ - \varrho_-\|_{L_T^2(L^\infty)} (\|\varrho_+\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c-1})} + \|\varrho_-\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c-1})}) \\
& \quad + \|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} (\|\varrho_+\|_{L_T^\infty(L^\infty)} + \|\varrho_-\|_{L_T^\infty(L^\infty)}) \\
& \lesssim \|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} (\|\varrho_+\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} + \|\varrho_-\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}). \quad (4.38)
\end{aligned}$$

Combining (4.33)-(4.38), we are led to the estimate

$$\begin{aligned}
& 2^{q(s_c-1)} \left(\|(\Delta_q \nabla \varrho_+, \Delta_q \nabla \varrho_-)\|_{L_T^2(L^2)} + \|\Delta_q \varrho_+ - \Delta_q \varrho_-\|_{L_T^2(L^2)} \right) \\
& \lesssim c_q (\|(\varrho_\pm, v_\pm)\|_{\tilde{L}_T^\infty(L^2)} + \|(\varrho_{\pm 0}, v_{\pm 0})\|_{B_{2,1}^{s_c}}) + c_q \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \\
& \quad + c_q \|(\varrho_\pm, v_\pm, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \left(\|(\nabla \varrho_\pm, \varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right). \quad (4.39)
\end{aligned}$$

Summing up (4.39) on $q \geq -1$, the inequality (4.24) is followed easily.

Next, we give the reason that $\varrho_+ - \varrho_- \in \tilde{L}_T^2(B_{2,1}^{s_c})$.

Claim 2. If $\varrho_+ - \varrho_- \in \tilde{L}_T^2(B_{2,1}^{s_c-1})$, $(\nabla \varrho_+, \nabla \varrho_-) \in \tilde{L}_T^2(B_{2,1}^{s_c-1})$, then

$$\varrho_+ - \varrho_- \in \tilde{L}_T^2(B_{2,1}^{s_c})$$

and

$$\|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \lesssim (\|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}). \quad (4.40)$$

Indeed, by virtue of the triangle inequality, one has

$$\|\nabla(\varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \lesssim \|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}, \quad (4.41)$$

which implies $\nabla(\varrho_+ - \varrho_-) \in \tilde{L}_T^2(B_{2,1}^{s_c-1})$, furthermore, it follows from the fact (4.22) that $\nabla(\varrho_+ - \varrho_-) \in \tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})$. According to Bernstein's inequality (Lemma 2.1), we obtain

$$\|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c})} \lesssim \|\nabla(\varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(\dot{B}_{2,1}^{s_c-1})} \lesssim \|\nabla(\varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}. \quad (4.42)$$

On the other hand, thanks to the embeddings

$$\tilde{L}_T^2(B_{2,1}^{s_c-1}) \hookrightarrow L_T^2(B_{2,1}^{s_c-1}) \hookrightarrow L_T^2(B_{2,2}^{s_c-1}) \hookrightarrow L_T^2(L^2),$$

we deduce that

$$\|\varrho_+ - \varrho_-\|_{L_T^2(L^2)} \lesssim \|\varrho_+ - \varrho_-\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}. \quad (4.43)$$

Then from the basic fact (4.22), the inequality (4.40) is achieved by (4.42) and (4.43) directly.

Finally, (4.23) follows from (4.24) and (4.40), which completes the proof of Lemma 4.2. \square

Lemma 4.3. *If $W \in \tilde{\mathcal{C}}_T(B_{2,1}^{s_c}) \cap \tilde{\mathcal{C}}_T^1(B_{2,1}^{s_c-1})$ is a solution of (3.2)-(3.3) for any $T > 0$, then the following estimate holds:*

$$\begin{aligned} & \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \\ & \lesssim \|(v_\pm, \tilde{E})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} + \|(v_{\pm 0}, \tilde{E}_0)\|_{B_{2,1}^{s_c}} + \left\{ \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right. \\ & \quad \left. + \|\nabla \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} + \|(\varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \sqrt{\|(\varrho_\pm, v_\pm, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}} \right. \\ & \quad \left. \times \left(\|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \right) \right\}, \end{aligned} \quad (4.44)$$

Proof. In fact, from (4.27) and (4.28), we have

$$\partial_t(v_+ - v_-) + (\nabla \varrho_+ - \nabla \varrho_-) + \frac{2}{\sqrt{\gamma}} \tilde{E} = f_2^+ - f_2^- - \frac{1}{\sqrt{\gamma}}(v_+ - v_-) \quad (4.45)$$

Applying the inhomogeneous localization operator Δ_q to (4.45), multiplying the resulting inequality by $\Delta_q \tilde{E}$ and integrating it over \mathbb{R}^N gives

$$\frac{d}{dt} \langle \Delta_q(v_+ - v_-), \Delta_q \tilde{E} \rangle + \frac{2}{\sqrt{\gamma}} \|\Delta_q \tilde{E}\|_{L^2}^2 = \sum_{i=1}^2 J_i(t), \quad (4.46)$$

where

$$\begin{aligned} J_1(t) : &= \langle \Delta_q(v_+ - v_-), \Delta_q \partial_t \tilde{E} \rangle \\ &= \frac{1}{\sqrt{\gamma}} \|\Delta_q(v_+ - v_-)\|_{L^2}^2 + \frac{1}{\sqrt{\gamma}} \langle \Delta_q(v_+ - v_-), \Delta_q(\nabla \times \tilde{B}) \rangle \\ &\quad + \frac{1}{\sqrt{\gamma}} \langle \Delta_q(v_+ - v_-), \Delta_q[\Phi(\varrho_+)v_+ + \varrho_+v_+] \rangle \\ &\quad - \frac{1}{\sqrt{\gamma}} \langle \Delta_q(v_+ - v_-), \Delta_q[\Phi(\varrho_-)v_- + \varrho_-v_-] \rangle \end{aligned}$$

and

$$\begin{aligned} J_2(t) : &= -\frac{1}{\sqrt{\gamma}} \langle \Delta_q(v_+ - v_-), \Delta_q \tilde{E} \rangle - \langle \Delta_q(\nabla \varrho_+ - \nabla \varrho_-), \Delta_q \tilde{E} \rangle \\ &\quad + \langle \Delta_q(f_2^+ - f_2^-), \Delta_q \tilde{E} \rangle. \end{aligned}$$

Through the straight but a little tedious calculations, with the aid of Propositions 2.1-2.2, we can obtain

$$\begin{aligned} &2^{2q(s_c-1)} \int_0^T |J_1(t)| dt \\ &\lesssim c_q^2 \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}^2 + c_q^2 \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \|\nabla \times \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} \\ &\quad + c_q^2 \|(\varrho_+, \varrho_-)\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c-1})} \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \end{aligned} \quad (4.47)$$

and

$$\begin{aligned} &2^{2q(s_c-1)} \int_0^T |J_2(t)| dt \\ &\lesssim c_q^2 \left(\|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|(\varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right) \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + c_q^2 \|(\varrho_\pm, v_\pm, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c-1})} \\ &\quad \times \left(\|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right) \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}. \end{aligned} \quad (4.48)$$

Then, combining with (4.46)-(4.48), we arrive at

$$\begin{aligned} &2^{2q(s_c-1)} \|\Delta_q \tilde{E}\|_{L_T^2(L^2)}^2 \\ &\lesssim c_q^2 \left(\|(v_+, v_-, \tilde{E})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}^2 + \|(v_{+0}, v_{-0}, \tilde{E}_0)\|_{B_{2,1}^{s_c}}^2 \right) + c_q^2 \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}^2 \\ &\quad + c_q^2 \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \|\nabla \times \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} + c_q^2 \left(\|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} \right. \\ &\quad \left. + \|(\varrho_+ - \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right) \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + c_q^2 \|(\varrho_+, \varrho_-)\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \\ &\quad + c_q^2 \|(\varrho_\pm, v_\pm, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \left(\|\nabla \varrho_\pm\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \|v_\pm\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right) \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}. \end{aligned} \quad (4.49)$$

Then employing Young's inequality and summing up the resulting inequality on $q \geq -1$ concludes the inequality (4.21). \square

Lemma 4.4. *If $W \in \tilde{C}_T(B_{2,1}^{s_c}) \cap \tilde{C}_T^1(B_{2,1}^{s_c-1})$ is a solution of (3.2)-(3.3) for any $T > 0$ and, then the following estimate holds:*

$$\begin{aligned} & \|\nabla \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} \\ \lesssim & \|(\tilde{E}, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} + \|(\tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}} + \left\{ \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right. \\ & \left. + \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \sqrt{\|\varrho_\pm\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}} \left(\|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|\nabla \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} \right) \right\}, \end{aligned} \quad (4.50)$$

Proof. Indeed, multiplying both sides of the fifth equation of (3.2) by $-\Delta_q(\nabla \times \tilde{B})$, taking integrations in $x \in \mathbb{R}^N$ and using integration by parts and replacing $\partial_t \Delta_q \tilde{B}$ from the fourth equation of (3.2), we arrive at

$$\begin{aligned} & -\frac{d}{dt} \langle \Delta_q(\nabla \times \tilde{E}), \Delta_q \tilde{B} \rangle + \frac{1}{\sqrt{\gamma}} \|\Delta_q(\nabla \times \tilde{B})\|_{L^2}^2 \\ = & \frac{1}{\sqrt{\gamma}} \|\Delta_q(\nabla \times \tilde{E})\|_{L^2}^2 - \frac{1}{\sqrt{\gamma}} \langle \Delta_q v_+, \Delta_q(\nabla \times \tilde{B}) \rangle + \frac{1}{\sqrt{\gamma}} \langle \Delta_q v_-, \Delta_q(\nabla \times \tilde{B}) \rangle \\ & - \frac{1}{\sqrt{\gamma}} \langle \Delta_q[\Phi(\varrho_+)v_+ + \varrho_+v_+], \Delta_q(\nabla \times \tilde{B}) \rangle, \end{aligned} \quad (4.51)$$

where we used the vector formula $\nabla \cdot (\vec{f} \times \vec{g}) = (\nabla \times \vec{f}) \cdot \vec{g} - (\nabla \times \vec{g}) \cdot \vec{f}$.

With the help of Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} & -\frac{d}{dt} \langle \Delta_q(\nabla \times \tilde{E}), \Delta_q \tilde{B} \rangle + \frac{1}{\sqrt{\gamma}} \|\Delta_q(\nabla \times \tilde{B})\|_{L^2}^2 \\ \leq & \frac{1}{\sqrt{\gamma}} \|\Delta_q(\nabla \times \tilde{E})\|_{L^2}^2 + \frac{1}{\sqrt{\gamma}} \|\Delta_q(v_+, v_-)\|_{L^2} \|\Delta_q(\nabla \times \tilde{B})\|_{L^2} \\ & + \frac{1}{\sqrt{\gamma}} \left(\|\Delta_q(\Phi(\varrho_+)v_+)\|_{L^2} + \|\Delta_q(\varrho_+v_+)\|_{L^2} \right) \|\Delta_q(\nabla \times \tilde{B})\|_{L^2} \\ & + \frac{1}{\sqrt{\gamma}} \left(\|\Delta_q(\Phi(\varrho_-)v_-)\|_{L^2} + \|\Delta_q(\varrho_-v_-)\|_{L^2} \right) \|\Delta_q(\nabla \times \tilde{B})\|_{L^2}. \end{aligned} \quad (4.52)$$

Note that the regularity of \tilde{E} in Lemma 4.4, we multiply (4.52) by the factor $2^{2q(s_c-2)}$ after integrating (4.52) with respect to $t \in [0, T]$ to get

$$\begin{aligned} & 2^{2q(s_c-2)} \|\Delta_q \nabla \tilde{B}\|_{L_T^2(L^2)}^2 \\ \lesssim & c_q^2 (\|\tilde{E}\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c-1})} \|\tilde{B}\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c-2})} + \|\tilde{E}_0\|_{B_{2,1}^{s_c-1}} \|\tilde{B}_0\|_{B_{2,1}^{s_c-2}}) \\ & + c_q^2 \left\{ \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}^2 + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \|\nabla \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} \right. \\ & \left. + \|(\varrho_+, \varrho_-)\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \|\nabla \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})} \right\}, \end{aligned} \quad (4.53)$$

where we notice the incompressible property of \tilde{B} and the elementary relation $\|\nabla \vec{f}\|_{L^2} \approx \|\nabla \cdot \vec{f}\|_{L^2} + \|\nabla \times \vec{f}\|_{L^2}$.

Furthermore, we apply Young's inequality to (4.53) and obtain

$$\begin{aligned}
& 2^{q(s_c-2)} \|\Delta_q \nabla \tilde{B}\|_{L_T^2(L^2)} \\
& \lesssim c_q (\|(\tilde{E}, \tilde{B})\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})} + \|(\tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}}) + c_q \left\{ \varepsilon \|\nabla \tilde{B}\|_{\tilde{L}_T(B_{2,1}^{s_c-2})} + \|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} \right. \\
& \quad \left. + \|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})} + \sqrt{\|(\varrho_+, \varrho_-)\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}} \left(\|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})} + \|\nabla \tilde{B}\|_{\tilde{L}_T(B_{2,1}^{s_c-2})} \right) \right\}, \quad (4.54)
\end{aligned}$$

where we take $\varepsilon \leq 1/2$.

Finally, after summing up (4.26) on $q \geq -1$, the desired inequality (4.50) is followed. \square

With the help of Lemmas 4.1-4.4, the inequality (4.2) in Proposition 4.1 follows, since we may introduce some positive constants to eliminate the terms $\|W\|_{\tilde{L}_T^\infty(B_{2,1}^{s_c})}$, $\|(\nabla \varrho_+, \nabla \varrho_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}$, $\|(v_+, v_-)\|_{\tilde{L}_T^2(B_{2,1}^{s_c})}$ and $\|\tilde{E}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-1})}$, $\|\nabla \tilde{B}\|_{\tilde{L}_T^2(B_{2,1}^{s_c-2})}$ arising in the right-hand sides of (4.3), (4.23), (4.44) and (4.50). See [23] for similar details, here, we omit them for brevity.

Having the Proposition 3.1 and Proposition 4.1, Theorem 1.2 (Global well-posedness) can be achieved by the standard boot-strap argument as in [13]. We give the outline of proof.

Proof of Theorem 1.2.

If the initial data satisfy $\|(\varrho_{\pm 0}, v_{\pm 0}, \tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}} \leq \frac{\delta_1}{2}$, by Proposition 3.1, then we can determine a time $T_1 > 0$ ($T_1 \leq T_0$) such that the local solutions of (3.2)-(3.3) exist in $\tilde{\mathcal{C}}_{T_1}(B_{2,1}^{s_c})$ satisfying $\|(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})\|_{\tilde{L}_{T_1}^\infty(B_{2,1}^{s_c})} \leq \delta_1$. Therefore from Proposition 4.1, the solutions satisfy the *a priori* estimate $\|(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})\|_{\tilde{L}_{T_1}^\infty(B_{2,1}^{s_c})} \leq C_1 \|(\varrho_{\pm 0}, v_{\pm 0}, \tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}} \leq \frac{\delta_1}{2}$, provided $\|(\varrho_{\pm 0}, v_{\pm 0}, \tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}} \leq \frac{\delta_1}{2C_1}$. So by Proposition 3.1 again, the system (3.2)-(3.3) for $t \geq T_1$ with the initial data $(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})(T_1)$ has a unique solution $(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})$ satisfying $\|(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})\|_{\tilde{L}_{(T_1, 2T_1)}^\infty(B_{2,1}^{s_c})} \leq \delta_1$, furthermore, $\|(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})\|_{\tilde{L}_{2T_1}^\infty(B_{2,1}^{s_c})} \leq \delta_1$. Then by Proposition 4.1 we have $\|(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})\|_{\tilde{L}_{2T_1}^\infty(B_{2,1}^{s_c})} \leq C_1 \|(\varrho_{\pm 0}, v_{\pm 0}, \tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}} \leq \frac{\delta_1}{2}$. Thus we can continuous the same process for $0 \leq t \leq nT_1, n = 3, 4, \dots$, and finally get a global solution $(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})$ satisfies

$$\begin{aligned}
& \|(\varrho_{\pm}, v_{\pm}, \tilde{E}, \tilde{B})\|_{\tilde{L}^\infty(B_{2,1}^{s_c})} \\
& + \mu_1 \left\{ \|(\varrho_+ - \varrho_-, v_{\pm})\|_{\tilde{L}^2(B_{2,1}^{s_c})} + \|(\nabla \varrho_+, \nabla \varrho_-, \tilde{E})\|_{\tilde{L}^2(B_{2,1}^{s_c-1})} + \|\nabla \tilde{B}\|_{\tilde{L}^2(B_{2,1}^{s_c-2})} \right\} \\
& \leq C_1 \|(\varrho_{\pm 0}, v_{\pm 0}, \tilde{E}_0, \tilde{B}_0)\|_{B_{2,1}^{s_c}} \leq \frac{\delta_1}{2}. \quad (4.55)
\end{aligned}$$

The choice of δ_1 is sufficient to ensure that $\frac{\gamma-1}{2}\varrho_{\pm} + 1 > 0$. Taking $\delta_0 = \min(\frac{\delta_1}{2}, \frac{\delta_1}{2C_1})$, then it follows from Remark 3.1 and the embedding properties (Lemma 2.2) that $(n_{\pm}, u_{\pm}, E, B) \in \mathcal{C}^1([0, \infty) \times \mathbb{R}^N)$ is a unique classical solution of (1.1)-(1.2) in the whole space.

For the periodic case, it suffice to prove the inequality (1.5). Recall that the definition of mean value \bar{f} in Theorem 1.2, we set $\bar{n}_{\pm 0} = 1$. Using the density equations in (1.1), we see that \bar{n}_{\pm} are conservative quantities for all time $t > 0$, so $\bar{n}_{\pm}(t) = 1$. From Poincaré inequality (see, e.g., [8]), we have

$$\|n_{\pm} - 1\|_{L_T^2(L^2(\mathbb{T}^N))} \lesssim \|\nabla n_{\pm}\|_{L_T^2(L^2(\mathbb{T}^N))} \lesssim \|\nabla n_{\pm}\|_{\tilde{L}^2(B_{2,1}^{s_c-1}(\mathbb{T}^N))}. \quad (4.56)$$

On the other hand, the Bernstein inequality (Lemma 2.1) implies

$$\|n_{\pm} - 1\|_{\tilde{L}^2(\dot{B}_{2,1}^{s_c}(\mathbb{T}^N))} \lesssim \|\nabla n_{\pm}\|_{\tilde{L}^2(\dot{B}_{2,1}^{s_c-1}(\mathbb{T}^N))}. \quad (4.57)$$

Apply the basic fact (4.22) again and get

$$\|n_{\pm} - 1\|_{\tilde{L}^2(\dot{B}_{2,1}^{s_c}(\mathbb{T}^N))} \lesssim \|\nabla n_{\pm}\|_{\tilde{L}^2(\dot{B}_{2,1}^{s_c-1}(\mathbb{T}^N))}. \quad (4.58)$$

Hence, (1.5) follows from (4.58) and (1.4) readily. This completes Theorem 1.2 eventually. \square

5 Appendix

In the last section, we present a remark on the continuity for compositions in Chemin-Lerner spaces. The corresponding stationary cases have been shown in [2] (see Corollary 2.66, P.97 and Corollary 2.91, P.105). Precisely, we have

Proposition 5.1. *Let $s > 0$, $1 \leq p, r, \theta, \theta_1, \theta_2, \theta_3, \theta_4 \leq \infty$, $F'' \in W_{loc}^{[s]+1, \infty}(I; \mathbb{R})$ with $F'(0) = 0$ and $T \in (0, \infty]$. Then*

$$\begin{aligned} & \|F(f) - F(g)\|_{\tilde{L}_T^{\theta}(B_{p,r}^s)} \\ & \lesssim (1 + \|f\|_{L_T^{\infty}(L^{\infty})} + \|g\|_{L_T^{\infty}(L^{\infty})})^{[s]+1} \|F''\|_{W^{[s]+1, \infty}} \left(\|f - g\|_{L_T^{\theta_1}(L^{\infty})} \right. \\ & \quad \left. \times \sup_{\kappa \in [0,1]} \|g + \kappa(f - g)\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)} + \|f - g\|_{\tilde{L}_T^{\theta_4}(B_{p,r}^s)} \sup_{\kappa \in [0,1]} \|g + \kappa(f - g)\|_{L_T^{\theta_3}(L^{\infty})} \right), \end{aligned} \quad (5.1)$$

where

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} = \frac{1}{\theta_3} + \frac{1}{\theta_4}.$$

Proof. Following from their suggestions in [2], we give the natural generalization. Note that the classical equality

$$F(f) - F(g) = (f - g) \int_0^1 F'(g + \kappa(f - g)) d\kappa, \quad (5.2)$$

it follows from Proposition 2.1 and 2.2 that

$$\begin{aligned} & \|F(f) - F(g)\|_{\tilde{L}_T^{\theta}(B_{p,r}^s)} \\ & \lesssim \|f - g\|_{L_T^{\theta_1}(L^{\infty})} \left\| \int_0^1 F'(g + \kappa(f - g)) d\kappa \right\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)} \\ & \quad + \left\| \int_0^1 F'(g + \kappa(f - g)) d\kappa \right\|_{L_T^{\theta_3}(L^{\infty})} \|f - g\|_{\tilde{L}_T^{\theta_4}(B_{p,r}^s)}, \end{aligned} \quad (5.3)$$

where

$$\left\| \int_0^1 F'(g + \kappa(f - g)) d\kappa \right\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)}$$

$$\begin{aligned}
&\leq \sup_{\kappa \in [0,1]} \|F'(g + \kappa(f - g))\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)} \\
&\lesssim \sup_{\kappa \in [0,1]} \left((1 + \|g + \kappa(f - g)\|_{L_T^\infty(L^\infty)})^{[s]+1} \|F''\|_{W^{[s]+1,\infty}} \|g + \kappa(f - g)\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)} \right) \\
&\lesssim (1 + \|f\|_{L_T^\infty(L^\infty)} + \|g\|_{L_T^\infty(L^\infty)})^{[s]+1} \|F''\|_{W^{[s]+1,\infty}} \sup_{\kappa \in [0,1]} \|g + \kappa(f - g)\|_{\tilde{L}_T^{\theta_2}(B_{p,r}^s)}, \quad (5.4)
\end{aligned}$$

and

$$\begin{aligned}
&\left\| \int_0^1 F'(g + \kappa(f - g)) d\kappa \right\|_{L_T^{\theta_3}(L^\infty)} \\
&\leq \sup_{\kappa \in [0,1]} \|F'(g + \kappa(f - g))\|_{L_T^{\theta_3}(L^\infty)} \\
&\lesssim \|F''\|_{L^\infty} \sup_{\kappa \in [0,1]} \|g + \kappa(f - g)\|_{L_T^{\theta_3}(L^\infty)}. \quad (5.5)
\end{aligned}$$

Therefore, (5.1) follows from (5.3), (5.4) and (5.5) readily. \square

Similarly, let us also mention the case in homogeneous Chemin-Lerner spaces.

Proposition 5.2. *Let $s > 0$, $1 \leq p, r, \theta, \theta_1, \theta_2, \theta_3, \theta_4 \leq \infty$, $F'' \in W_{loc}^{[s]+1,\infty}(I; \mathbb{R})$ with $F'(0) = 0$ and $T \in (0, \infty]$. Besides, let $s < N/p$ or $s = N/p$ and $r = 1$. Then*

$$\begin{aligned}
&\|F(f) - F(g)\|_{\tilde{L}_T^\theta(\dot{B}_{p,r}^s)} \\
&\lesssim (1 + \|f\|_{L_T^\infty(L^\infty)} + \|g\|_{L_T^\infty(L^\infty)})^{[s]+1} \|F''\|_{W^{[s]+1,\infty}} \left(\|f - g\|_{L_T^{\theta_1}(L^\infty)} \right. \\
&\quad \times \left. \sup_{\kappa \in [0,1]} \|g + \kappa(f - g)\|_{\tilde{L}_T^{\theta_2}(\dot{B}_{p,r}^s)} + \|f - g\|_{\tilde{L}_T^{\theta_4}(\dot{B}_{p,r}^s)} \sup_{\kappa \in [0,1]} \|g + \kappa(f - g)\|_{L_T^{\theta_3}(L^\infty)} \right), \quad (5.6)
\end{aligned}$$

where

$$\frac{1}{\theta} = \frac{1}{\theta_1} + \frac{1}{\theta_2} = \frac{1}{\theta_3} + \frac{1}{\theta_4}.$$

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